The spherical Hall algebra of $\overline{\mathrm{Spec}(\mathcal{O}_K)}$

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Abstract

We generalize the result of M. Kapranov, O. Schiffmann, and E. Vasserot [KSV12] by showing that, for a number field K with class number 1, the spherical Hall algebra of $\overline{\mathrm{Spec}(\mathcal{O}_K)}$, where $\mathcal{O}_K$ is the ring of integers of K, is isomorphic to the Paley-Wiener shuffle algebra associated to a Hecke L-function corresponding to K.

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0 Introduction

M. Kapranov, O. Schiffmann, and E. Vasserot showed in [KSV12] that the spherical Hall algebra of $\overline{\mathrm{Spec}(\mathbb{Z})}$ is isomorphic to the Paley-Wiener shuffle algebra associated to the global Harish-Chandra function $\Phi(s)$, defined as $\Phi(s) = \frac{\zeta^*(s)}{\zeta^*(s+1)}$, where $\zeta^*(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ is the completed zeta function.

In order to define the Hall algebra of $\overline{\mathrm{Spec}(\mathbb{Z})}$, *vector bundles on* $\overline{\mathrm{Spec}(\mathbb{Z})}$ are defined as triples $E = (L, V, q)$ where $V = \mathbb{R}^n$, $L \subset V$ is a $\mathbb{Z}$-lattice of maximal rank, and q is a positive-definite quadratic form. The *rank of* E is defined as $\mathrm{rk}(E) = \dim_{\mathbb{R}}(V) = \mathrm{rk}_{\mathbb{Z}}(L)$. These vector bundles form a category $\mathcal{B}un$. The Hall algebra of $\overline{\mathrm{Spec}(\mathbb{Z})}$ is defined as $H = \bigoplus_{n\geqslant 0} H_n$, where $H_n = C_c^\infty(\mathrm{Bun}_n)$ and Bun_n consists of the isomorphism classes of rank n vector bundles. For this, it is shown that Bun_n can be identified with $\mathrm{GL}_n(\mathbb{Z})\setminus\mathrm{GL}_n(\mathbb{R})/O_n$ and $\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})/O_n\times\prod_p \mathrm{GL}_n(\mathbb{Z}_p)$, and that it has a C^∞-orbifold structure, and a *Hall product* $*$ is constructed and proven to make H into a graded associative algebra with unit 1. The *spherical Hall algebra* SH is defined as the subalgebra generated by $H_1 \subset H$. It is shown that we can identify H_1 with $C_c^\infty(\mathbb{R}_+)$, so SH can be seen as a subset of $\bigoplus_{n\geqslant 0} C_c^\infty(\mathbb{R}_+^n)$.

On the other hand, $\bigoplus_{n\geqslant 0}\mathcal{M}er(\mathbb{C}^n)$ can be made into a graded associative algebra with unit 1 via the *shuffle product* Ⓢ, defined as

$$F \,\text{Ⓢ}\, F' = \sum_{w\in \mathrm{Sh}(m,n)} w(F\otimes F')\cdot \Phi_{w^{-1}}.$$

Here, $\mathrm{Sh}(m,n)$ is the set of (m,n)-shuffles, and $\Phi_{w^{-1}}(s_1,\dots,s_{m+n}) = \prod_{i<j \text{ and } w(i)>w(j)} \Phi(s_{w(i)}-s_{w(j)})$. The *Paley-Wiener shuffle algebra associated to* Φ is denoted by $\mathcal{SH}(\Phi)_{\mathcal{PW}}$ and is defined as the subalgebra generated by the subspace $\mathcal{PW}(\mathbb{C}) \subset \bigoplus_{n\geqslant 0}\mathcal{M}er(\mathbb{C}^n)$ of Paley-Wiener functions.

The main result of [KSV12] shows that $SH \cong \mathcal{SH}(\Phi)_{\mathcal{PW}}$.

Theorem 0.1. *The Mellin transform* $\mathcal{M} : SH_1 = C_c^\infty(\mathbb{R}_+) \xrightarrow{\sim} \mathcal{PW}(\mathbb{C})$ *extends to an isomorphism of algebras* $SH \to \mathcal{SH}(\Phi)_{\mathcal{PW}}$.

The explicit isomorphism $\mathrm{Ch} : SH \xrightarrow{\sim} \mathcal{SH}(\Phi)_{\mathcal{PW}}$ is defined as $\mathrm{Ch} = \mathcal{M}\circ\widetilde{\mathrm{CT}}$, where $\mathcal{M}$ is the *Mellin transform* $\mathcal{M} : C^\infty(\mathbb{R}_+^n) \to \mathcal{PW}(\mathbb{C}^n)$ given by $\mathcal{M}(f)(s) = \int_{a\in\mathbb{R}_+^n} f(a)a^s\frac{\mathrm{d}a}{a}$, and $\widetilde{\mathrm{CT}} : C_c^\infty(\mathbb{R}_+^n)\to C^\infty(\mathbb{R}_+^n)$ is a twisted version of the *constant term map* $\mathrm{CT} : H_n = C_c^\infty(\mathrm{GL}_n(\mathbb{Z})\backslash\mathrm{GL}_n(\mathbb{R})/O_n) \to C^\infty(\mathbb{R}_+^n)$, which takes an automorphic form on GL_n and yields its constant term.

Note that, in particular, $\mathbb{Z}$ is the ring of integers of the field $\mathbb{Q}$. In our paper, we generalize this result for a number field K with class number 1, that is, such that its ring of integers $\mathcal{O}_K$ is a principal ideal domain. We follow [KSV12] closely.

We consider the real and complex Archimedean places $\nu\in\mathcal{S}$ of K, which correspond to embeddings $\nu : K\to\mathbb{C}$. From this, we generalize $\mathbb{Z}\subset\mathbb{R}$ to $\mathcal{O}_K \subset R = \prod_{\nu\in\mathcal{S}} R_\nu$, where $R_\nu = \mathbb{R}$ if $\nu\in\mathcal{S}_\mathbb{R}$ and $R_\nu = \mathbb{C}$ if $\nu\in\mathcal{S}_\mathbb{C}$, and the embedding is given by $a\mapsto(\nu(a))_{\nu\in\mathcal{S}}$. In our definition of vector bundles over $\overline{\mathrm{Spec}(\mathcal{O}_K)}$, $E = (L,V,q)$, where $V = R^n$, $L\subset R^n$ is an $\mathcal{O}_K$-lattice of maximal rank, and $q = \prod_\nu q_\nu$, where q_ν is a positive-definite quadratic form if $\nu\in\mathcal{S}_\mathbb{R}$ and a positive-definite Hermitian form if $\nu\in\mathcal{S}_\mathbb{C}$. From this, we can define Bun_n, which will now be identified with

$$\mathrm{GL}_n(\mathcal{O}_K)\setminus\mathrm{GL}_n(R)/\mathcal{K}_n = \mathrm{GL}_n(K)\setminus\mathrm{GL}_n(\mathbb{A}_K)/\widehat{\mathcal{K}_n},$$

where $\mathcal{K}_n = \prod_{\nu\in\mathcal{S}} O_\nu$ and $\widehat{\mathcal{K}_n} = \prod_{\nu\in\mathcal{S}} O_\nu\times\prod_\mathfrak{p}\mathrm{GL}_n(\mathcal{O}_{K,\mathfrak{p}})$. Here, O_ν is the orthogonal group for $\nu\in\mathcal{S}_\mathbb{R}$ and the unitary group for $\nu\in\mathcal{S}_\mathbb{C}$, and $\mathfrak{p}$ goes over the prime ideals of $\mathcal{O}_K$ with $\mathcal{O}_{K,\mathfrak{p}}$ being the $\mathfrak{p}$-adic integers. In the same way as before, we define the Hall algebra of $\overline{\mathrm{Spec}(\mathcal{O}_K)}$, and the spherical Hall algebra SH is again defined as the subalgebra generated by $H_1\subset H$. However, this time we will be able to identify SH as a subset of $\bigoplus_{n\geqslant 0} C_c^\infty(\mathcal{B}^n)$, where $\mathcal{B} = \mathrm{Bun}_1 = \mathcal{O}_K^\times\setminus\prod_{\nu\in\mathcal{S}}\mathbb{R}_+^\times \cong \mathcal{O}_K^\times\setminus\prod_{\nu\in\mathcal{S}}\mathbb{R}$ via taking logarithms. We know (say, from [Mil20]) that $\Lambda = \mathcal{O}_K^\times$ is (non-canonically) isomorphic to $\mathbb{Z}^{r_1+r_2-1}$, where $r_1 = \#\mathcal{S}_\mathbb{R}$ and $r_2 = \#\mathcal{S}_\mathbb{C}$ are the number of real and complex places, respectively. From this,

$$\mathcal{B} = \mathcal{O}_K^\times\setminus\prod_{\nu\in\mathcal{S}}\mathbb{R}_+^\times \cong (\mathbb{R}^{r_1+r_2-1}/\Lambda)\times\mathbb{R}_+ \cong (\mathbb{R}/\mathbb{Z})^{r_1+r_2-1}\times\mathbb{R}_+.$$

As a result, our Fourier transform will now have the form $\mathcal{F} : C_c^\infty(\mathcal{B}^n)\to\mathcal{PW}((\Lambda^*\times\mathbb{C})^n)$, where

$$\mathcal{F}f(\lambda^*,s) = \int_{x\in D} f(x)|x|_\mathcal{B}^s x^{2\pi i\lambda^*}\frac{\mathrm{d}x}{|x|_\mathcal{B}}.$$

Here, $|x|_\mathcal{B} = \prod_{i=1}^{r_1} x_i\prod_{j=r_1+1}^{r_1+r_2} x_j^2$ and D is a fundamental domain for $\mathcal{B}^n$. We modify the constant term map $\widetilde{\mathrm{CT}} : C_c^\infty(\mathcal{B}^n)\to C^\infty(\mathcal{B}^n)$ and the homomorphism $\mathrm{Ch} = \mathcal{F}\circ\widetilde{\mathrm{CT}}$ accordingly. Instead of getting $\zeta(s)$ in our computations, we end up getting a Hecke L-function corresponding to K (for the definition and properties of this function, see [Neu99]), which we call $L_K(\lambda^*,s)$, and we define $\Phi_K(\lambda^*,s)$ accordingly. After making $\bigoplus_{n\geqslant 0}\mathcal{M}er((\Lambda^*\times\mathbb{C})^n)$ into an algebra via Ⓢ and defining the *Paley-Wiener shuffle algebra with respect to*

$\Phi_K(\lambda^*, s)$, which we denote by $\mathcal{SH}(\Phi_K)_{\mathcal{PW}}$, as the subalgebra generated by $\mathcal{PW}(\Lambda^* \times \mathbb{C})$, we prove the following main result.

Theorem 0.2. *The map* $\mathrm{Ch} : SH \to \mathcal{SH}(\Phi_K)_{\mathcal{PW}}$ *is an isomorphism of algebras.*

Structure of the paper

In section 1, we define vector bundles on $\overline{\mathrm{Spec}(\mathcal{O}_K)}$, morphisms between them, their rank, and their degree. We prove that we have bijections

$$\mathrm{Bun}_n = \mathrm{GL}_n(\mathcal{O}_K)\backslash \mathrm{GL}_n(R)/\mathcal{K}_n = \mathrm{GL}_n(K)\backslash \mathrm{GL}_n(\mathbb{A}_K)/\widehat{\mathcal{K}_n}$$

Afterwards, we define *subbundles* $E' \subset E$, which we will use for our definition of Hall product. In section 2, we define the Hall product $* : H_m \otimes_{\mathbb{C}} H_n \to H_{m+n}$ as

$$(f * g)(E) = \sum_{\substack{E' \subset E \\ \mathrm{rk}(E')=m}} \deg(E')^{\frac{n}{2}} \deg(E/E')^{-\frac{m}{2}} \cdot f(E')g(E/E')$$

and we show that it makes $H = \bigoplus_{n=0}^{\infty} H_n$ into a graded associative algebra with unit 1. We define the spherical Hall algebra SH.

In section 3, we define the set $\mathcal{M}er((\Lambda^* \times \mathbb{C})^n)$ of meromorphic functions on $(\Lambda^* \times \mathbb{C})^n$, and we define $L_K(\lambda^*, s)$, a *Hecke L-function corresponding to K*. Then, we define $\Phi_K(s, \lambda^*)$ as $\Phi_K(\lambda^*, s) = \frac{L_K^*(\lambda^*,s)}{L_K^*(\lambda^*,s+1)}$, where $L_K^*(\lambda^*, s) = \pi^{-(r_1/2)s}(2\pi)^{-r_2 s}\Gamma_K(\lambda^*, s)L_K(\lambda^*, s)$. In section 4, using $\Phi_K(\lambda^*, s)$, we define the shuffle multiplication ⓢ, which makes $\bigoplus_{n\geqslant 0} \mathcal{M}er((\Lambda^* \times \mathbb{C})^n)$ into a graded associative algebra with unit 1. We then define the subalgebra $\mathcal{SH}(\Phi_K)_{\mathcal{PW}}$.

In section 5, we define Fourier transform $\mathcal{F}$ and its inverse transform $\mathcal{G}$. In section 6, we define $M_w : C_c^\infty(\mathcal{B}^n) \to C^\infty(\mathcal{B}^n)$, the principal series intertwiner, as

$$(M_w\varphi)(g) = \int_{u \in U_w(\mathbb{A}_K)\backslash U(\mathbb{A}_K)} \varphi(wug)\,\mathrm{d}u.$$

Here, w is a permutation and $U_w = U \cap (w^{-1}Uw)$. This operator will allow us to connect the Hall product, the Fourier transform, and the constant term, which we will define next.

In section 7, we introduce the *constant term* $\mathrm{CT}_n : H_n \to C^\infty(\mathcal{B}^n)$ as

$$\mathrm{CT}_n(a_1, \dots, a_n) = \int_{u\in U_n(\mathcal{O}_K)\backslash U_n(R)} f(u \cdot \mathrm{diag}(a_1, \dots, a_n))\,\mathrm{d}u = \int_{u_{\mathbb{A}}\in U_n(K)\backslash U_n(\mathbb{A}_K)} f(u_{\mathbb{A}} \cdot \mathrm{diag}(a_1, \dots, a_n))\,\mathrm{d}u_{\mathbb{A}}$$

and its twist $\widetilde{\mathrm{CT}}$ as $\widetilde{\mathrm{CT}}(f)(a) = \mathrm{CT}(f)(a) \cdot \delta(a)^{\frac{1}{2}}$, where $\delta(a)$ is the Iwasawa Jacobian. We will use these to construct the map $\mathrm{Ch}_n = \mathcal{F} \circ \widetilde{\mathrm{CT}}_n$. We will show that $\mathrm{Ch} : SH \to \mathcal{SH}(\Phi_K)_{\mathcal{PW}}$ is an isomorphism.

Notation

We use $\mathbb{A}_K$ to denote the adelic ring of the number field K and $\mathcal{N}$ to denote the ideal norm. We use z to denote a tuple $((\lambda_1^*, s_1), \dots, (\lambda_n^*, s_n)) \in (\Lambda^* \times \mathbb{C})^n$. To denote the set $\{1, \dots, n\}$ by $[\![1, n]\!]$, and we use $\#S$ to denote the cardinality of the set S. We write A_n for the subgroup of diagonal matrices in GL_n.

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1 Vector bundles on $\overline{\mathrm{Spec}(\mathcal{O}_K)}$

Let K be a number field such that its ring of integers $\mathcal{O}_K$ is a principal ideal domain. In other words, the class number of K is 1. Let $\mathcal{S}$ be the set of real and complex Archimedean places of K. Recall that a place $\nu \in \mathcal{S}$ corresponds to an embedding $\nu : K \to \mathbb{C}$ and a norm $|\cdot|_\nu : \nu(K) \to \mathbb{R}_+$. Let $\mathcal{S}_\mathbb{R}$ and $\mathcal{S}_\mathbb{C}$ denote the sets of real and complex places, respectively, and let $r_1 = \#\mathcal{S}_\mathbb{R}$ and $r_2 = \#\mathcal{S}_\mathbb{C}$. Let $R = \prod_{\nu\in\mathcal{S}} R_\nu$, where $R_\nu = \mathbb{R}$ for $\nu \in \mathcal{S}_\mathbb{R}$ and $R_\nu = \mathbb{C}$ for $\nu \in \mathcal{S}_\mathbb{C}$.

A *vector bundle of rank n over* $\overline{\mathrm{Spec}(\mathcal{O}_K)}$ is a triple $E = (L, V, q)$, where $V = R^n$, $L \subset V$ is an $\mathcal{O}_K$-lattice of maximal rank (i.e., a free $\mathcal{O}_K$-module of rank n such the map $L \otimes_{\mathcal{O}_K} R \to V$ induced by the inclusion $L \subset V$ is an isomorphism of R-modules), and $q = \prod_{\nu\in\mathcal{S}} q_\nu$, where the q_ν is positive-definite quadratic form on $\mathbb{R}^n$ for $\nu \in \mathcal{S}_\mathbb{R}$ and a positive-definite Hermitian form on $\mathbb{C}^n$ for $\nu \in \mathcal{S}_\mathbb{C}$. The *rank* of a vector bundle $E = (L, V, q)$ is defined as $\mathrm{rk}(E) = \mathrm{rk}_R V = \mathrm{rk}_{\mathcal{O}_K} L$. For example, if $K = \mathbb{Q}(i)$, we have the *trivial bundle* $(\mathbb{Z}[i], \mathbb{C}, z\overline{z})$, and if $K = \mathbb{Q}(\sqrt{2})$, we have a vector bundle $(\{(a + b\sqrt{2}, a - b\sqrt{2}) : a, b \in \mathbb{Z}\}, \mathbb{R}\times\mathbb{R}, x^2 \times 3x^2)$. Both of these examples have rank 1.

A *morphism* of vector bundles $f : E = (L, V, q) \to E' = (L', V', q')$ is an R-linear map $f : V \to V'$ such that $f(L) \subset L'$ and $q'_\nu(f(v)) \leqslant q_\nu(v)$ for all $v \in V$ and $\nu \in \mathcal{S}$. We say that $f : E \to E'$ is an *isomorphism* if $f : V \to V'$ is an R-isomorphism, $f(L) = L'$, and $q'(f(v)) = q(v)$ for all $v \in V$. As a result, we get a category $\mathcal{B}un$ where the objects are vector bundles and the morphisms are as previously defined.

1.1 Isomorphism classes of rank n vector bundles

Let us consider the class of isomorphism classes of rank n vector bundles, which we denote by Bun_n. To be able to define functions in Bun_n, we show that we can identify Bun_n in the following two ways:

$$\mathrm{Bun}_n \simeq \mathrm{GL}_n(\mathcal{O}_K)\backslash\mathrm{GL}_n(R)/\mathcal{K}_n \simeq \mathrm{GL}_n(K)\backslash\mathrm{GL}_n(\mathbb{A}_K)/\widehat{\mathcal{K}_n}.$$

Here, $\mathcal{K}_n = \prod_{\nu\in\mathcal{S}} O_\nu$, where O_ν is the orthogonal group O_n for $\nu \in \mathcal{S}_\mathbb{R}$ and the unitary group U_n for $\nu \in \mathcal{S}_\mathbb{C}$, and $\widehat{\mathcal{K}_n} = \prod_{\nu\in\mathcal{S}} O_\nu \times \prod_{\mathfrak{p}\in\mathrm{Spec}\,\mathcal{O}_K} \mathrm{GL}_n(\mathcal{O}_{K,\mathfrak{p}})$, where $\mathcal{O}_{K,\mathfrak{p}}$ denotes the set of $\mathfrak{p}$-adic integers. The inclusion $\mathrm{GL}_n(\mathcal{O}_K) \subset \mathrm{GL}_n(R)$ is induced from the embedding $\mathcal{O}_K \subset R$ given by $a \mapsto (\nu(a))_{\nu\in\mathcal{S}}$, and the other inclusions come from the natural embeddings as well.

Before proving this statement, let us first exhibit some lemmas.

Lemma 1.1. *For $(p) = \mathfrak{p} \in \mathrm{Spec}\,\mathcal{O}_K$, we have*

$$\mathrm{GL}_n(K_\mathfrak{p}) = \mathrm{GL}_n(\mathcal{O}_K)A_n(\{p^\mathbb{Z}\})\mathrm{GL}_n(\mathcal{O}_{K,\mathfrak{p}}).$$

Proof. Pick any $g \in \mathrm{GL}_n(K_\mathfrak{p})$. We split into several steps.

Step 1: after left and right multiply by matrices in $\mathrm{GL}_n(\mathbb{Z})$ we can assume g_{11} has the smallest $v_\mathfrak{p}$ value.

Step 2: after right multiply by a matrix in $\mathrm{GL}_n(\mathcal{O}_{K,\mathfrak{p}})$ we can assume $g_{1j} = 0$ for $j > 1$. Note that g_{11} still has the smallest $v_\mathfrak{p}$ value.

Step 3: repeat Step 1 and 2 to the $(n-1)\times(n-1)$ submatrix (deleting the first row and column) we can assume $g_{2j} = 0$ for $j > 2$; note that g_{11} still has the smallest $v_\mathfrak{p}$ value, while g_{22} has the smallest $v_\mathfrak{p}$ value in the submatrix.

Step 4: repeat Step 3 to submatrices we can assume g is lower-triangular, with g_{ii} having the smallest $v_\mathfrak{p}$ value in the submatrix $(g_{\ell j})_{\ell,j\geqslant i}$.

Step 5: after right multiply by a matrix in $\mathrm{GL}_n(\mathcal{O}_{K,\mathfrak{p}})$ we can assume $g_{ii} = p^{a_i}$ for some $a_1 < a_2 < \cdots < a_n$.

Step 6: after left multiply by a matrix in $\mathrm{GL}_n(\mathcal{O}_K)$ we can assume $v_\mathfrak{p}(g_{i1}) \geqslant a_n$.

Step 7: repeat Step 6 to the submatrices we can assume $v_\mathfrak{p}(g_{ij}) \geqslant a_n$ for $i > j$.

Step 8: finally, after right multiply a matrix in $\mathrm{GL}_n(\mathcal{O}_{K,\mathfrak{p}})$ we can assume $g = \mathrm{diag}\,(p^{a_1}, \ldots, p^{a_n})$, which finishes the proof. $\square$

Corollary 1.2. *We have*

$$\mathrm{GL}_n\left(\mathbb{A}_K^f\right) = \mathrm{GL}_n(K)\mathrm{GL}_n\left(\widehat{\mathcal{O}}_K\right),$$

where $\mathbb{A}_K^f$ is the ring of finite adeles in K and $\widehat{\mathcal{O}}_K$ is the profinite completion of $\mathcal{O}_K$, so that $\mathrm{GL}_n(\widehat{\mathcal{O}}_K) = \prod_{\mathfrak{p}\in\mathrm{Spec}(\mathcal{O}_K)} \mathrm{GL}_n(\mathcal{O}_{K,\mathfrak{p}})$.

Proof. Pick any $g = (g_\mathfrak{p})_{\mathfrak{p}\in\mathrm{Spec}(\mathcal{O}_K)} \in \mathrm{GL}_n(\mathbb{A}_K^f)$, where $g_\mathfrak{p} \in \mathrm{GL}_n(\mathcal{O}_{K,\mathfrak{p}})$ for all $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_K) - \{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$ being a finite set. Let $(p_i) = \mathfrak{p}_i$. We then do the following procedure.

Step 1: from Lemma 1.1 we can find $u_1 \in \mathrm{GL}_n(\mathcal{O}_K[1/p_1])$ such that $u_1 g_{\mathfrak{p}_1} \in \mathrm{GL}_n(\mathcal{O}_{K,\mathfrak{p}_1})$.

Step 2: from Lemma 1.1 we can find $u_2 \in \mathrm{GL}_n(\mathcal{O}_K[1/p_2])$ such that $u_2 u_1 g_{\mathfrak{p}_2} \in \mathrm{GL}_n(\mathcal{O}_{K,\mathfrak{p}_2})$.

$\vdots$

Step m: from Lemma 1.1 we can find $u_m \in \mathrm{GL}_n(\mathcal{O}_K[1/p_m])$ such that $u_m u_{m-1} \cdots u_1 g_{\mathfrak{p}_m} \in \mathrm{GL}_n(\mathcal{O}_{K,\mathfrak{p}_m})$.

Now put $u = u_m u_{m-1} \cdots u_1 \in \mathrm{GL}_n(K)$, then we have $u g_{\mathfrak{p}_i} \in \mathrm{GL}_n(\mathcal{O}_{K,\mathfrak{p}_i})$ for $1 \leqslant i \leqslant m$ because $\mathcal{O}_K[1/p_i] \subset \mathcal{O}_{K,\mathfrak{p}_j}$ for $i \neq j$. For the same reason we have $u g_\mathfrak{p} \in \mathrm{GL}_n(\mathcal{O}_{K,\mathfrak{p}})$ for all $p \in \mathrm{Spec}(\mathcal{O}_K) - \{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$. Thus, we have $g = u^{-1}(u g_\mathfrak{p})_{\mathfrak{p}\in\mathrm{Spec}(\mathcal{O}_K)} \in \mathrm{GL}_n(K)\mathrm{GL}_n(\widehat{\mathcal{O}}_K)$, which finishes the proof. ☐

Now we are at a position to prove the two identifications of Bun_n.

Proposition 1.3. *We have bijections*

$$\mathrm{Bun}_n = \mathrm{GL}_n(\mathcal{O}_K)\backslash\mathrm{GL}_n(R)/\mathcal{K}_n = GL_n(K)\backslash GL_n(\mathbb{A}_K)/\widehat{\mathcal{K}_n}.$$

Proof. First, let us show that $\mathrm{Bun}_n = \mathrm{GL}_n(\mathcal{O}_K)\backslash\mathrm{GL}_n(R)/\mathcal{K}_n$. Let $(M_*q)(x) = q(Mx)$ for any matrix M and any quadratic form q. If $q_{\mathrm{st}}(x) = x^T x$, $q(x) = x^T A^T A x = A_* q_{\mathrm{st}}$ (as any symmetric matrix may be written in the form $A^T A$), and $B \in GL_n(R)$ is a matrix such that $B(L) = \mathcal{O}_K$. Consider the maps $\alpha : \mathrm{GL}_n(\mathcal{O}_K)\backslash\mathrm{GL}_n(R)/\mathcal{K}_n \to \mathrm{Bun}_n$ and $\beta : \mathrm{Bun}_n \to \mathrm{GL}_n(\mathcal{O}_K)\backslash\mathrm{GL}_n(R)/\mathcal{K}_n$ defined by

$$\alpha : g \mapsto (\mathcal{O}_K^n, R^n, (g^{-1})_*(q_{\mathrm{st}}))$$

and

$$\beta : (L, R^n, q) \overset{B}{\cong} (\mathcal{O}_K^n, R^n, (AB^{-1})_*(q_{\mathrm{st}})) \mapsto BA^{-1}$$

To check that α is well-defined, if we choose the representative hgk instead, note that

$$(\mathcal{O}_K^n, R^n, ((hgk)^{-1})_*(q_{\mathrm{st}})) \overset{h^{-1}}{\cong} (\mathcal{O}_K^n, R^n, ((gk)^{-1})_*(q_{\mathrm{st}})) = (\mathcal{O}_K^n, R^n, (g^{-1})_*(q_{\mathrm{st}}))$$

To see that β is well-defined, if $(L, R^n, q) \overset{f}{\cong} (L', R^n, q')$, we have that $B'f = B$ and $f_*q' = q \implies A'f = A$ (both equalities in the double coset), so it follows that $BA^{-1} = B'A'^{-1}$ in the double coset.

Finally, to check these are inverses, note that

$$(\alpha \circ \beta)(L, R^n, q) = \alpha(BA^{-1}) = (\mathcal{O}_K^n, R^n, (AB^{-1})_*(q_{\mathrm{st}})) \cong (L, R^n, q)$$

and

$$(\beta \circ \alpha)(g) = \beta(\mathcal{O}_k^n, R^n, (q^{-1})_*(q_{\mathrm{st}})) = g$$

as desired. Now, let us show that $\mathrm{GL}_n(\mathcal{O}_K)\backslash\mathrm{GL}_n(R)/\mathcal{K}_n = GL_n(K)\backslash GL_n(\mathbb{A}_K)/\widehat{\mathcal{K}_n}$.

Now we will show that the natural map

$$\mathrm{GL}_n(\mathcal{O}_K)\backslash\mathrm{GL}_n(R)/\mathcal{K}_n \to GL_n(K)\backslash GL_n(\mathbb{A}_K)/\widehat{\mathcal{K}_n}$$

induced from the inclusion $\mathrm{GL}_n(R) \hookrightarrow GL_n(\mathbb{A}_K)$ is a bijection. It is surjective from Corollary 1.2. Now we show the injectivity.

Let us say that $\overline{(g_\nu)_{\nu\in\mathcal{S}}}$ has the same image as $\overline{(g'_\nu)_{\nu\in\mathcal{S}}}$. Then there exists $u \in \mathrm{GL}_n(K)$ and $\big((h_\mathfrak{p})_{\mathfrak{p}\in\mathrm{Spec}(\mathcal{O}_K)}, (h_\nu)_{\nu\in\mathcal{S}}\big) \in K_n = \prod_{\mathfrak{p}\in\mathrm{Spec}(\mathcal{O}_K)} \mathrm{GL}_n(\mathcal{O}_{K,\mathfrak{p}}) \times \prod_{\nu\in\mathcal{S}} O_\nu$ such that $\big((\mathrm{id})_{\mathfrak{p}\in\mathbb{P}}, (g'_\nu)_{\nu\in\mathcal{S}}\big) = u^{-1}\big((\mathrm{id})_{\mathfrak{p}\in\mathrm{Spec}(\mathcal{O}_K)}, (g_\nu)_{\nu\in\mathcal{S}}\big)\big((h_\mathfrak{p})_{\mathfrak{p}\in\mathrm{Spec}(\mathcal{O}_K)}, (h_\nu)_{\nu\in\mathcal{S}}\big)$, that is, $h_\mathfrak{p} = u$ and $g'_\nu = u^{-1} g_\nu h_\nu$.

However, then $u \in \mathrm{GL}_n(K) \cap \bigcap_{\mathfrak{p}\in\mathrm{Spec}(\mathcal{O}_K)} \mathrm{GL}_n(\mathcal{O}_{K,\mathfrak{p}}) = \mathrm{GL}_n(\mathcal{O}_K)$ since $\mathcal{O}_K = K \cap \bigcap_{\mathfrak{p}\in\mathrm{Spec}(\mathcal{O}_K)} \mathcal{O}_{K,\mathfrak{p}}$ and $\mathcal{O}_K^\times = K^\times \cap \bigcap_{\mathfrak{p}\in\mathrm{Spec}(\mathcal{O}_K)} \mathcal{O}_{K,\mathfrak{p}}^\times$, and so $\overline{(g_\nu)_{\nu\in\mathcal{S}}} = \overline{(g'_\nu)_{\nu\in\mathcal{S}}}$, which concludes the proof. $\square$

From now on, when we say a vector bundle $E \in \mathrm{Bun}_n$, we will actually mean the isomorphism class of E.

Now, let us compute Bun_1 explicitly. As the orthogonal and unitary groups are just $\{\pm 1\}$ and S_1 for $n = 1$, respectively, we have that

$$\begin{aligned}
\mathrm{Bun}_1 &= \mathrm{GL}_1(\mathcal{O}_K)\backslash \mathrm{GL}_1(R)/\prod_{\nu\in\mathcal{S}}(O_\nu)_1 \\
&= \mathcal{O}_K^\times \backslash R^\times / \prod_{\nu\in\mathcal{S}_\mathbb{R}} \{\pm 1\} \times \prod_{\nu'\in\mathcal{S}_\mathbb{C}} S_1 \\
&= \mathcal{O}_K^\times \backslash \prod_{\nu\in\mathcal{S}_\mathbb{R}} \mathbb{R}^\times \times \prod_{\nu'\in\mathcal{S}_\mathbb{C}} \mathbb{C}^\times / \prod_{\nu\in\mathcal{S}_\mathbb{R}} \{\pm 1\} \times \prod_{\nu'\in\mathcal{S}_\mathbb{C}} S_1 \\
&= \mathcal{O}_K^\times \backslash \prod_{\nu\in\mathcal{S}_\mathbb{R}} \mathbb{R}^\times/\{\pm 1\} \times \prod_{\nu'\in\mathcal{S}_\mathbb{C}} \mathbb{C}^\times/S_1 \\
&= \mathcal{O}_K^\times \backslash \prod_{\nu\in\mathcal{S}} \mathbb{R}_+^\times.
\end{aligned}$$

Here, $\mathcal{O}_K^\times$ embeds into $\prod_{\nu\in\mathcal{S}} \mathbb{R}_+^\times$ by $a \mapsto (|\nu(a)|_\nu)_{\nu\in\mathcal{S}}$. Furthermore, by [Mil20], $\mathcal{O}_K^\times$ is a $\mathbb{Z}$-lattice of rank $r_1 + r_2 - 1$, so if we denote this lattice by Λ, we have that

$$\mathrm{Bun}_1 = \mathcal{O}_K^\times \backslash \prod_{\nu\in\mathcal{S}} \mathbb{R}_+^\times = (\mathbb{R}^{r_1+r_2-1}/\Lambda) \times \mathbb{R}_+^\times \cong (\mathbb{R}/\mathbb{Z})^{r_1+r_2-1} \times \mathbb{R}_+^\times$$

where we used the isomorphism $\log : \mathbb{R}_+^\times \xrightarrow{\sim} \mathbb{R}$. However, the isomorphism $\mathbb{R}^{r_1+r_2-1}/\Lambda \cong (\mathbb{R}/\mathbb{Z})^{r_1+r_2-1}$ is not canonical, so we prefer to use $\mathbb{R}^{r_1+r_2-1}/\Lambda$ instead of $(\mathbb{R}/\mathbb{Z})^{r_1+r_2-1}$. This identification of Bun_1 will allow us to define functions on it, as well as a Fourier transform for these functions. Denote $\mathcal{B} = \mathrm{Bun}_1 = \mathcal{O}_K^\times \backslash \prod_{\nu\in\mathcal{S}} \mathbb{R}_+^\times = (\mathbb{R}^{r_1+r_2-1}/\Lambda) \times \mathbb{R}_+^\times$ for the rest of the paper. We can write an element $a \in \mathcal{B}$ as $\overline{(a_i)}_{i=1}^{r_1+r_2}$, where $a_i \in \mathbb{R}_+^\times$ and we take the equivalence class after taking the quotient by $\mathcal{O}_K$. We define a norm $|\cdot|_\mathcal{B} : \mathcal{B} \to \mathbb{R}_+$ by

$$|a|_\mathcal{B} = \prod_{i=1}^{r_1} a_i \prod_{j=r_1+1}^{r_1+r_2} a_j^2.$$

The complex norms are squared because we have two embeddings ν and $\overline{\nu}$ for each complex place ν. Note that this norm is well-defined, as for $b \in \mathcal{O}_K^\times$, we have that $|b|_\mathcal{B} = \big|(|\nu(b)|)_{\nu\in\mathcal{S}}\big|_\mathcal{B} = \prod_{\nu\in\mathcal{S}_\mathbb{R}} |\nu(b)|_\nu \prod_{\nu\in\mathcal{S}_\mathbb{C}} |\nu(b)|_\nu^2 = 1$.

Let β be the map $\beta : \mathrm{Bun}_n \to \mathrm{GL}_n(\mathcal{O}_K)\backslash \mathrm{GL}_n(R)/\mathcal{K}_n$. We define the *degree of* $E \in \mathrm{Bun}_n$ as $\deg(E) = |\beta(E)|_\mathcal{B}$. Note that we can take the norm of $\beta(E) \in R^\times$ because we have a natural map

$$R^\times \to \mathcal{O}_K^\times \backslash R^\times / \prod_{\nu\in\mathcal{S}_\mathbb{R}} \{\pm 1\} \times \prod_{\nu'\in\mathcal{S}_\mathbb{C}} S_1 = \mathcal{B}.$$

We can check that deg is well-defined as $\det(h), \det(k) \in \mathcal{O}_K$ for $h \in \mathrm{GL}_n(\mathcal{O}_K)$ and $k \in \mathcal{K}_n$.

1.2 Subbundles

We say that

$$0 \longrightarrow E' \xrightarrow{i} E \xrightarrow{j} E'' \longrightarrow 0$$

is a *short exact sequence in $\mathcal{B}un$* if we have the following:

1. The induced sequences of modules are short exact.
2. The form q' is equal to $i^*(q)$, where $(i^*q)(v') = q(i(v'))$, $v' \in V'$.
3. The form q'' is equal to $j_*(q)$, where $(j_*q)(v'') = \min_{j(v)=v''} q(v)$, $v'' \in V''$.

We say that i is an *admissible monomorphism*. We define a *subbundle in E* as the equivalence class of admissible monomorphisms $E' \to E$ modulo isomorphisms of the source. For each subbundle E', we then have a quotient bundle $E/E' = E''$.

We have the following three propositions.

Proposition 1.4. *Let $E = (L, V, q)$ be a vector bundle on $\overline{\mathrm{Spec}(\mathcal{O}_K)}$. We have a bijection between each of the following sets:*

1. *Rank r subbundles $E' \subset E$.*
2. *Rank r primitive sublattices, that is, submodules $L' \subset L$ such that L/L' has no torsion.*
3. *K-linear subspaces $W' \subset L \otimes_{\mathcal{O}_K} K$ of dimension r.*

Proof. Given a rank r subbundle $E' = (L', V', q')$, we can map it to a lattice L'. This lattice will be primitive because there must be some E'' such that $0 \to L' \to L \to L'' \to 0$ is exact, so $L/L' = L''$ would have to be free. This gives a bijection between (1) and (2). For a bijection between (1) and (3), we can map E' to $L' \otimes_{\mathcal{O}_K} K$. We can check that these maps are indeed bijections, as desired. □

Proposition 1.5. *Let $E = (L, V, q)$ be a vector bundle on $\overline{\mathrm{Spec}(\mathcal{O}_K)}$. For any $r \in \mathbb{Z}_+$ and $a \in \mathbb{R}_+$, the set of subbundles $E' \subset E$ with $\mathrm{rk}(E') = r$ and $\deg(E) \geqslant a$ is finite.*

Proof. The proof is the same as in [KSV12]. □

Proposition 1.6. *The set $\mathrm{Ext}^1(E'', E')$ has a natural structure isomorphic to $(R/\mathcal{O}_K)^{n'n''}$, where $n' = \mathrm{rk}(E')$ and $n'' = \mathrm{rk}(E'')$.*

Proof. The proof is the same as in [KSV12]. □

2 Hall algebra

To be able to define functions in Bun_n, we show the following.

Proposition 2.1. *The set Bun_n can be endowed with an orbifold structure.*

Proof. The proof is very similar to that in [KSV12]. □

As a result, we can consider the set $H_n = C_c^\infty(\mathrm{Bun}_n)$ of compactly supported functions in Bun_n. Let $H = \bigoplus_{n \geqslant 0} H_n$ be the *Hall algebra of* $\overline{\mathrm{Spec}(\mathcal{O}_K)}$. However, it is still not clear that H has an algebra structure. Our next step is to show this.

Given $f \in H_m$ and $g \in H_n$, we define the *Hall product* $f * g \in H_{m+n}$ as

$$(f * g)(E) = \sum_{\substack{E' \subset E \\ \mathrm{rk}(E') = m}} \deg(E')^{\frac{n}{2}} \deg(E/E')^{-\frac{m}{2}} \cdot f(E') g(E/E')$$

Proposition 2.2. *$f * g$ is well-defined and makes H into a graded associative algebra.*

Proof. The proof is very similar to that in [KSV12]. □

Now, let SH be the *spherical Hall algebra of* $\overline{\mathrm{Spec}(\mathcal{O}_K)}$, which we define as the subalgebra $SH \subset H$ generated by H_1. In other words, $SH = \bigoplus_{n\geqslant 0} H_1^{\otimes n}$. We define the map $*_{1^n} : H_1^{\otimes n} \to H_n$ as the *multiplication map*, that is,

$$\varphi_1 \otimes \cdots \otimes \varphi_n \mapsto \varphi_1 * \cdots * \varphi_n.$$

Translating the Hall product into group-theoretic terms, we have the following proposition.

Proposition 2.3. *$f' * f''$ can be rewritten as*

$$(f' * f'')(g) = \sum_{\gamma \in P_{n',n''}(K)\backslash \mathrm{GL}_n(K)} f(\gamma g)$$

where $P_{n',n''}$ is the parabolic subgroup of GL_n, that is, the group of block lower triangular matrices, $g = (g', g'')$ and $f(\gamma g) = |\det(\gamma g')|^{\frac{n''}{2}} \cdot |\det(\gamma g'')|^{-\frac{n'}{2}} \cdot f'(\gamma g') f''(\gamma g'')$. This makes sense, as we have an isomorphism

$$(\mathrm{GL}_n n'(K)\backslash \mathrm{GL}_{n'}(\mathbb{A}_K)/\widehat{\mathcal{K}_{n'}}) \times (\mathrm{GL}_{n''}(K)\backslash \mathrm{GL}_{n''}(\mathbb{A}_K)/\widehat{\mathcal{K}_{n''}}) \xrightarrow{\sim} (U_{n',n''}(\mathbb{A}_K) A_{n',n''}(K))\backslash \mathrm{GL}_n(\mathbb{A}_K)/\widehat{\mathcal{K}_n}$$

coming from the Iwasawa decomposition.

3 Hecke L-function associated to K

Let H be the hyperplane in $\mathbb{R}^{r_1+r_2} \cong \mathbb{R}_+^{r_1+r_2}$ that is in the kernel of $|\cdot|$, and let Λ be the lattice $\mathcal{O}_K^\times$ in H. Define dual lattice $\Lambda^* = \{\lambda^* \in H^* : \langle \lambda^*, \lambda \rangle \in \mathbb{Z} \text{ for all } \lambda \in \Lambda\}$, which is isomorphic to $\mathbb{Z}^{r_1+r_2-1}$ as $\mathbb{Z}$-modules. Note that for we can define group homomorphism $\langle \lambda^*, \cdot \rangle : \mathbb{R}^{r_1+r_2} \to \mathbb{R}$ to be the composition of the projection $\mathbb{R}^{r_1+r_2} \to H$ and $\langle \lambda^*, \cdot \rangle : H \to \mathbb{R}$.

For $\lambda^* \in \Lambda^*, s \in \mathbb{C}$ such that $\mathrm{Re}\, s > 1$, we define

$$L_K(\lambda^*, s) = \sum_{(x) \subset \mathcal{O}_K} \frac{1}{\exp(2\pi i \langle \lambda^*, \log|x| \rangle) \mathcal{N}(x)^s},$$

where the summation runs through all ideals in $\mathcal{O}_K$ and $\mathcal{N}$ denote the norm in K. Here by $\log|x|$ we mean the tuple $(\log|\nu(x)|)_{\nu \in \mathcal{S}}$ in $\mathbb{R}^{r_1+r_2}$.

The Euler product of L_K reads

$$L_K(\lambda^*, s) = \prod_{\mathfrak{p} \subset \mathcal{O}_K} \left(1 - \exp(-2\pi i \langle \lambda^*, \log|p| \rangle) \mathcal{N}(p)^{-s}\right)^{-1},$$

where the summation runs through all prime ideals in $\mathcal{O}_K$ and p is (any) generator for $\mathfrak{p}$.

We use $\mathcal{M}er((\Lambda^*)^n \times \mathbb{C}^n)$ to denote the set of meromorphic functions whose domain is $(\Lambda^*)^n \times \mathbb{C}^n$, that is, those functions $f : (\Lambda^*)^n \times \mathbb{C}^n \to \mathbb{C}$ such that for each $\lambda^* \in (\Lambda^*)^n$, the function $f(\lambda^*, \cdot) : \mathbb{C}^n \to \mathbb{C}$ is a meromorphic function. Also, we use $\mathcal{O}((\Lambda^*)^n \times \mathbb{C}^n)$ to denote the set of entire functions whose domain is $(\Lambda^*)^n \times \mathbb{C}^n$.

Now we let

$$\begin{aligned}
\Gamma_K(\lambda^*, s) &= \prod_{\nu \in \mathcal{S}_\mathbb{R}} \Gamma(s/2 + \pi i \lambda_\nu^*) \prod_{\nu \in \mathcal{S}_\mathbb{C}} \Gamma(s + \pi i \lambda_\nu^*) \\
L_K^*(\lambda^*, s) &= \pi^{-(r_1/2)s} (2\pi)^{-r_2 s} \Gamma_K(\lambda^*, s) L_K(\lambda^*, s) \\
\Phi_K(\lambda^*, s) &= \frac{L_K^*(\lambda^*, s)}{L_K^*(\lambda^*, s+1)}.
\end{aligned}$$

From [Neu99] we see we can extend L_K^* to a meromorphic function in $\Lambda^* \times \mathbb{C}$, which satisfies the functional equation $L_K^*(-\lambda^*, 1-s) = L_K^*(\lambda^*, s)$. Thus, we have

$$\begin{aligned}\Phi_K(-\lambda^*, -s) &= \frac{L_K^*(-\lambda^*, -s)}{L_K^*(-\lambda^*, -s+1)} \\ &= \frac{L_K^*(\lambda^*, s+1)}{L_K^*(\lambda^*, s)} \\ &= \Phi_K(\lambda^*, s)^{-1}.\end{aligned}$$

Finally, a *Paley-Wiener function* in $\Lambda^* \times \mathbb{C}$ is defined as follows. For any $\lambda^* \in \Lambda^*$ there exists a unique $\lambda^\vee \in H$ such that $\langle \lambda^*, x\rangle = \lambda^\vee \cdot x$ for any $x \in H$, where $\cdot$ is the standard dot product on $H \subset \mathbb{R}^{r_1+r_2}$. Then the set of Paley-Wiener fucntions, denoted by $\mathcal{PW}(\Lambda^* \times \mathbb{C})$, are functions in $\mathcal{O}(\Lambda^* \times \mathbb{C})$ such that there exists $B \in \mathbb{R}_+$ with the property that for any $N \in \mathbb{Z}^+$, we have

$$|f(\lambda^*, s)| = O\big((1+|s|)^{-N}(1+|\Lambda^\vee|)^{-N}\exp(B|\mathrm{Re}\, s|)\big).$$

4 Shuffle algebra associated to Φ_K

Let $\mathfrak{S}_n$ denote the symmetric group of permutations of $[\![1, n]\!]$. For this paper, we have the following conventions:

- For any n-tuple $s = (s_1, \ldots, s_n)$, we denote $w(s) = (s_{w^{-1}(1)}, \ldots, s_{w^{-1}(n)})$, so that $(ww')(s) = w(w'(s))$.
- We embed $\mathfrak{S}_n$ in GL_n via $w \mapsto \begin{pmatrix} e_{w(1)} & \ldots & e_{w(n)}\end{pmatrix}$, where e_i is the column vector whose i-th entry is 1 and its other entries are 0. Note that this embedding is an injective group homomorphism.

For $w \in \mathfrak{S}_n$, let

$$\Phi_{K,w}(\lambda^*, s) = \prod_{\substack{1\leqslant i<j\leqslant n \\ w(i)>w(j)}} \Phi(\lambda_j^* - \lambda_i^*, s_j - s_i)$$

We have the following proposition, which will be helpful later.

Proposition 4.1. *We have that* $\Phi_{K,w}(\lambda^*, s) = \Phi_{K,w^{-1}}(-w(\lambda^*), -w(s))$.

Proof. Note that

$$
\begin{aligned}
\Phi_{K,w^{-1}}(-w(\lambda^*), -w(s)) &= \prod_{\substack{1\leqslant i<j\leqslant n\\ w^{-1}(i)>w^{-1}(j)}} \Phi_K((-w(\lambda^*))_i - (-w(\lambda^*))_j, (-w(s))_i - (-w(s))_j) \\
&= \prod_{\substack{1\leqslant i<j\leqslant n\\ w^{-1}(i)>w^{-1}(j)}} \Phi_K(w(\lambda^*)_j - w(\lambda^*)_i, w(s)_j - w(s)_i) \\
&= \prod_{\substack{1\leqslant i<j\leqslant n\\ w^{-1}(i)>w^{-1}(j)}} \Phi_K(\lambda^*_{w^{-1}j} - \lambda^*_{w^{-1}(i)}, s_{w^{-1}j} - s_{w^{-1}(i)}) \\
&= \prod_{\substack{1\leqslant w(i)<w(j)\leqslant n\\ i>j}} \Phi_K(\lambda^*_j - \lambda^*_i, s_j - s_i) \\
&= \prod_{\substack{1\leqslant j<i\leqslant n\\ w(j)>w(i)}} \Phi_K(\lambda^*_j - \lambda^*_i, s_j - s_i) \\
&= \prod_{\substack{1\leqslant i<j\leqslant n\\ w(i)>w(j)}} \Phi_K(\lambda^*_i - \lambda^*_j, s_i - s_j) \\
&= \Phi_{K,w}(s)
\end{aligned}
$$

as desired. ☐

We define the set of (m,n)-*shuffles* as the set $\mathrm{Sh}(m,n)$ of permutations $w \in \mathfrak{S}_{m+n}$ such that $w(i) < w(j)$ whenever $i,j \in [\![1,m]\!]$ or $i,j \in [\![m+1, m+n]\!]$. For example, $\mathrm{Sh}(2,1) = \{\mathrm{id}, (2\ 3), (1\ 2\ 3)\}$. It is straightforward to verify that $\#\mathrm{Sh}(m,n) = \binom{m+n}{m}$.

Now, let us define the *shuffle multiplication* $Ⓢ_{m,n} : \mathcal{M}er((\Lambda^* \times \mathbb{C})^m) \otimes_{\mathbb{C}} \mathcal{M}er((\Lambda^* \times \mathbb{C})^n) \to \mathcal{M}er((\mathbb{C}\times\Lambda)^{m+n})$ by $F \otimes F' \mapsto F Ⓢ F'$, where

$$
(F Ⓢ F')(z_1, \ldots, z_{m+n}) = \sum_{w\in\mathrm{Sh}(m,n)} F(z_{w(1)}, \ldots, z_{w(m)}) F'(z_{w(m+1)}, \ldots, z_{w(m+n)}) \cdot \Phi_{K,w^{-1}}(z_1, \ldots, z_{m+n}).
$$

Proposition 4.2. *The shuffle multiplication makes* $\bigoplus_{n\geqslant 0} \mathcal{M}er((\Lambda^* \times \mathbb{C})^n)$ *into a graded associative algebra with unit 1.*

Proof. The proof can be found in [FO95]. ☐

Let $\mathcal{SH}(\Phi_K)_{\mathcal{PW}}$ be the *Paley-Wiener shuffle algebra associated to* Φ_K, which we define as the subalgebra in $\left(\bigoplus_{n\geqslant 0} \mathcal{M}er((\Lambda^* \times \mathbb{C})^n), Ⓢ\right)$ generated by the subspace $\mathcal{PW}(\Lambda^* \times \mathbb{C})$.

5 Fourier transform

We define the Fourier transform to be an operator $\mathcal{F} : \mathcal{PW}((\Lambda^* \times \mathbb{C})^n) \to C_c^\infty(\mathcal{B}^n)$ as

$$
\mathcal{F}f(\lambda^*, s) = \int_{x\in D} f(x)|x|^s_{\mathcal{B}} x^{2\pi i\lambda^*} \frac{\mathrm{d}x}{|x|_{\mathcal{B}}},
$$

where $D \subset \mathbb{R}_+^{(r_1+r_2)n}$ is a fundamental domain for $\mathcal{B}^n = (\mathcal{O}_K^\times)^n \setminus \mathbb{R}_+^{(r_1+r_2)n}$.

Similarly, we define the inverse Fourier transform to be an operator $\mathcal{G} : \mathcal{PW}((\Lambda^* \times \mathbb{C})^n) \to C_c^\infty(\mathcal{B}^n)$ as

$$\mathcal{G}g(r) = \frac{1}{(2\pi i)^{(r_1+2r_2)n}} \sum_{\lambda^* \in (\Lambda^*)^n} \int_{s \in \sigma_0 + i\mathbb{R}^n} g(\lambda^*, s)|r|_{\mathcal{B}}^{-s} r^{-2\pi i \lambda^*} \, \mathrm{d}s.$$

Here, σ_0 is chosen such that the integral converges.

It can be checked that $\mathcal{F}\mathcal{G}(g) = g$ and $\mathcal{G}\mathcal{F}(f) = f$. We also have the following generalization of a classical result.

Proposition 5.1. *The Fourier transform $\mathcal{F} : H_1 = C_c^\infty(\mathcal{B}) \to \mathcal{PW}(\Lambda^* \times \mathbb{C})$ is an isomorphism.*

Proof. The proof can be adapted from Vol. II, Thm. IX.11 in [RS75]. $\square$

6 The principal series intertwiners

We define

$$\begin{aligned} &\mathscr{C} : \Lambda^* \times \mathbb{C} \to C^\infty(\mathcal{B}) \\ &\mathscr{C}(\lambda^*, s)(a) = |a|_{\mathcal{B}}^s a^{2\pi i \lambda^*} = |a|_{\mathcal{B}}^s \exp(2\pi i \langle \lambda^*, \log a \rangle)). \end{aligned}$$

For $w \in \mathfrak{S}_n$, we define $U_w = U \cap (w^{-1} U w)$ and

$$\begin{aligned} \overline{M}_w : C_c^\infty(U(\mathbb{A}_K)A_n(K)\backslash \mathrm{GL}_n(\mathbb{A}_K)/\widehat{\mathcal{K}}_n) &\to C^\infty(U(\mathbb{A}_K)A_n(K)\backslash \mathrm{GL}_n(\mathbb{A}_K)/\widehat{\mathcal{K}}_n) \\ \overline{M}_w(\psi)(g) &= \int_{u \in U_w(\mathbb{A}_K)\backslash U(\mathbb{A}_K)} \psi(wug) \, \mathrm{d}u, \end{aligned}$$

where we are using the Tamagawa measure on $\mathbb{A}_K$.

We will see that $\mathrm{diag} : \mathcal{B}^n \to U(\mathbb{A}_K)A_n(K)\backslash \mathrm{GL}_n(\mathbb{A}_K)/\widehat{\mathcal{K}}_n$ is a bijection, and we use a twist to identify their smooth functions together:

$$\begin{aligned} \delta^{1/2} : C^\infty(\mathcal{B}^n) &\xrightarrow{\sim} C^\infty(U(\mathbb{A}_K)A_n(K)\backslash \mathrm{GL}_n(\mathbb{A}_K)/\mathcal{K}_n) \\ \varphi &\mapsto \left[\mathrm{diag}(a) \mapsto \varphi(a)\delta(a)^{1/2} \right], \end{aligned}$$

where $\delta(a) = \prod_{1 \leqslant i < j \leqslant n} \frac{|a_j|_{\mathcal{B}}}{|a_i|_{\mathcal{B}}}$.

The fact that $\mathrm{diag} : \mathcal{B}^n \to U(\mathbb{A}_K)A_n(K)\backslash \mathrm{GL}_n(\mathbb{A}_K)/\widehat{\mathcal{K}}_n$ is an isomorphism also gives us the following proposition, which can be proved by translating the Hall product formula into group-theoretical terms.

Proposition 6.1. *If $\varphi \in H_1^{\otimes n}$, $g \in \mathcal{B}^n$, and $\widetilde{\varphi}(g) = \varphi(\mathrm{diag}^{-1}(g)) \cdot \delta(g)^{-\frac{1}{2}}$, then*

$$(*_{1^n}(\varphi))(g) = \sum_{\gamma \in B_n(\mathcal{O}_K)\backslash \mathrm{GL}_n(\mathcal{O}_K)} \widetilde{\varphi}(\gamma g).$$

Finally, we define

$$M_w : C_c^\infty(\mathcal{B}^n) \to C^\infty(\mathcal{B}^n)$$

to be the composition

$$C_c^\infty(\mathcal{B}^n) \xrightarrow{\delta^{1/2}} C_c^\infty(U(\mathbb{A}_K)A_n(K)\backslash \mathrm{GL}_n(\mathbb{A}_K)/\mathcal{K}_n) \xrightarrow{\overline{M}_w} C^\infty(U(\mathbb{A}_K)A_n(K)\backslash \mathrm{GL}_n(\mathbb{A}_K)/\mathcal{K}_n) \xrightarrow{\delta^{-1/2}} C^\infty(\mathcal{B}^n).$$

Note that we can also apply M_w to functions in $C^\infty(\mathcal{B}^n)$ that decay fast enough.

One of our main goals is to compute

$$M_w\mathscr{C}(\lambda^*, s) = M_w\left(\bigotimes_{i=1}^n \mathscr{C}(\lambda_i^*, s_i)\right)$$

for $\lambda^* \in (\Lambda^*)^n, s \in \mathbb{C}^n$.

6.1 The diagonal map is bijective

In this subsection, we will prove the following proposition.

Proposition 6.2. *The natural embedding $\mathbb{R}_+^{r_1+r_2} \hookrightarrow \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \hookrightarrow \mathbb{A}_K$ induces a bijection between sets*

$$\begin{aligned} \mathcal{B}^n &\xrightarrow{\sim} U_n(\mathbb{A}_K)A_n(K)\backslash \mathrm{GL}_n(\mathbb{A}_K)/K_n \\ \overline{(r_1, \ldots, r_n)} &\mapsto \overline{((\mathrm{id})_{p\in\mathbb{P}_K}, \mathrm{diag}\,(r_1, \ldots, r_n))}, \end{aligned}$$

where $r_i \in \prod_{\nu\in\mathcal{S}} \mathbb{R}_+$.

To prove this proposition, we need a few lemmas.

Lemma 6.3. *For prime ideal $(p) = \mathfrak{p} \subset \mathcal{O}_K$, we have*

$$\mathrm{GL}_n(K_\mathfrak{p}) = B_n(\mathcal{O}_K[1/p])\mathrm{GL}_n(\mathcal{O}_{K,\mathfrak{p}}),$$

where $B_n = U_nA_n$ is the set of lower-triangular matrices.

Proof. Pick $g \in \mathrm{GL}_n(K_\mathfrak{p})$. We do the following procedure.

1. After right multiply by a matrix in $\mathrm{GL}_n(\mathcal{O}_{K,\mathfrak{p}})$, we can assume $v_\mathfrak{p}(g_{11}) \leqslant v_\mathfrak{p}(g_{1j})$ for all $j \in [\![1, n]\!]$.
2. After right multiply by a matrix in $\mathrm{GL}_n(\mathcal{O}_{K,\mathfrak{p}})$, we can assume $g_{1j} = 0$ for $j > 1$.
3. Repeat Step 1 and 2, we can assume $g \in U_n(K_\mathfrak{p})$.
4. After right multiply by a matrix in $\mathrm{GL}_n(\mathcal{O}_{K,\mathfrak{p}})$, we can assume $g_{ii} = p^{a_i}$ for some $a_i \in \mathbb{Z}$.
5. After left multiply by a diagonal matrix in $B_n(\mathcal{O}_K[1/p])$, we can assume $g_{ii} = 1$ for $i \in [\![1, n]\!]$.
6. After left multiply by a matrix in $U_n(\mathcal{O}_K[1/p])$, we can assume $v_\mathfrak{p}(g_{i1}) \geqslant 0$ for $i > 1$.
7. Repeat Step 6, we can assume $g_{ij} \in \mathcal{O}_{K,\mathfrak{p}}$ for $i > j$. Now the resulting matrix is in $\mathrm{GL}_n(\mathcal{O}_{K,\mathfrak{p}})$ and we finishes our proof.

□

Lemma 6.4. *We have a bijection between sets*

$$\begin{aligned} \mathbb{R}_+^n &\xrightarrow{\sim} U_n(\mathbb{R})\backslash \mathrm{GL}_n(\mathbb{R})/O_n \\ (r_1, \ldots, r_n) &\mapsto \overline{\mathrm{diag}\,(r_1, \ldots, r_n)}. \end{aligned}$$

Proof. To see the injectivity, let us say that $(r_1, \ldots, r_n)$ and $(r_1', \ldots, r_n')$ have the same image, i.e., there exist $u \in U_n(\mathbb{R}), o \in O_n$ such that $\mathrm{diag}\,(r_1', \ldots, r_n') = u\,\mathrm{diag}\,(r_1, \ldots, r_n)o$. Then take their image under the map $A \mapsto AA^T$ gives $\mathrm{diag}\left((r_1')^2, \ldots, (r_n')^2\right) = u\,\mathrm{diag}\left(r_1^2, \ldots, r_n^2\right)u^T$, or $u^{-1}\,\mathrm{diag}\left((r_1')^2, \ldots, (r_n')^2\right) = \mathrm{diag}\left(r_1^2, \ldots, r_n^2\right)u^T$. However, LHS is lower-triangular, while RHS is upper-triangular, so we see both are diagonal, i.e., $u = \mathrm{id}$, which implies $\mathrm{diag}\,(r_1, \ldots, r_n) = \mathrm{diag}\,(r_1', \ldots, r_n')$.

Now we show the surjectivity. We identify $\mathrm{GL}_n(\mathbb{R})/O_n$ as the space of positive definite symmetric forms, say X, via the map $A \mapsto AA^T$. Then the action of $U_n(\mathbb{R})$ on X is $u \cdot q = uqu^T$. If the matrix of q corresponds to the basis $\{v_1, \ldots, v_n\}$, then the matrix uqu^T corresponds to applying q to the basis $u(v_1\ v_2\ \cdots\ v_n)^T$, and we see we can always use such actions to make the basis to be orthogonal, which shows the original map is surjective. □

Lemma 6.5. *Let $\mathscr{U}_n$ be the set of unitary matrices in $\mathrm{GL}_n(\mathbb{C})$, then we have a bijection between sets*

$$\begin{aligned}\mathbb{R}_+^n &\xrightarrow{\sim} U_n(\mathbb{C})\backslash\mathrm{GL}_n(\mathbb{C})/\mathscr{U}_n\\ (r_1,\dots,r_n) &\mapsto \overline{\mathrm{diag}\,(r_1,\dots,r_n)}.\end{aligned}$$

Proof. Just mimic of Lemma 6.4. ☐

Now we are at a position to prove Proposition 6.2.

Proof of Proposition 6.2. The map is clearly well-defined. We first show that it is surjective.

Pick any $\bar{g} = \overline{\big((g_\mathfrak{p})_{\mathfrak{p}\in\mathrm{Spec}(\mathcal{O}_K)}, (g_\nu)_{\nu\in\mathcal{S}}\big)} \in U_n(\mathbb{A}_K)A_n(K)\backslash\mathrm{GL}_n(\mathbb{A}_K)/K_n$, where $g_\mathfrak{p} \in \mathrm{GL}_n(\mathcal{O}_{K,\mathfrak{p}})$ for $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_K) - \{\mathfrak{p}_1,\dots,\mathfrak{p}_m\}$ for some primes $\mathfrak{p}_1,\dots,\mathfrak{p}_m$. For $\mathfrak{p} \in \{\mathfrak{p}_1,\dots,\mathfrak{p}_m\}$, from Lemma 6.3 we can write $g_\mathfrak{p} = u_\mathfrak{p}a_\mathfrak{p}h_\mathfrak{p} \in U_n(\mathcal{O}_K[1/p])A_n(\mathcal{O}_K[1/p])\mathrm{GL}_n(\mathcal{O}_{K,\mathfrak{p}})$ (where $(p) = \mathfrak{p}$). Also, from Lemma 6.4 and Lemma 6.5, for each $\nu \in \mathcal{S}$, we can write $g_\nu = u_\nu a_\nu o_\nu \in U_n(K_\nu)A_n(\mathbb{R}_+)O_\nu$. Now take $a = \prod_{\mathfrak{p}\in\{\mathfrak{p}_1,\dots,\mathfrak{p}_m\}} a_\mathfrak{p} \in A_n(K)$, then

$$\begin{aligned}\bar{g} &= \overline{a^{-1}g}\\ &= \overline{\left(\left((a^{-1}u_\mathfrak{p}a)\left(\prod_{\mathfrak{p}'\in\mathbb{P}'-\{\mathfrak{p}\}} a_{\mathfrak{p}'}^{-1}\right)h_\mathfrak{p}\right)_{\mathfrak{p}\in\{\mathfrak{p}_1,\dots,\mathfrak{p}_m\}}, (a^{-1}g_\mathfrak{p})_{\mathfrak{p}\in\mathrm{Spec}(\mathcal{O}_K)-\{\mathfrak{p}_1,\dots,\mathfrak{p}_m\}}, ((a^{-1}u_\nu a)a^{-1}a_\nu o_\nu)_{\nu\in\mathcal{S}}\right)}\\ &= \overline{\big((\mathrm{id})_{\mathfrak{p}\in\mathrm{Spec}(\mathcal{O}_K)}, (a^{-1}a_\nu)_{\nu\in\mathcal{S}}\big)},\end{aligned}$$

which is in the image of the map.

Now we show that the map is injective. To do this, define map $\varphi : K^\times \to \prod_{\nu\in\mathcal{S}}\mathbb{R}_+, x \mapsto |\nu(x)|$ and assume $\overline{(r_1,\dots,r_n)}$ has the same image as $\overline{(r_1',\dots,r_n')}$, where $r_i, r_i' \in \prod_{\nu\in\mathcal{S}}\mathbb{R}_+$. Then there exists

$$\big((u_\mathfrak{p})_{\mathfrak{p}\in\mathrm{Spec}(\mathcal{O}_K)}, (u_\nu)_{\nu\in\mathcal{S}}\big) \in U_n(\mathbb{A}_K), \mathrm{diag}\,(x_1,\dots,x_n) \in A_n(K), \big((h_\mathfrak{p})_{\mathfrak{p}\in\mathrm{Spec}(\mathcal{O}_K)}, (h_\nu)_{\nu\in\mathcal{S}}\big) \in K_n$$

such that

$$\big((\mathrm{id})_{\mathfrak{p}\in\mathrm{Spec}(\mathcal{O}_K)}, \mathrm{diag}\,(r_1',\dots,r_n')\big) = \big((u_\mathfrak{p})_{\mathfrak{p}\in\mathrm{Spec}(\mathcal{O}_K)}, (u_\nu)_{\nu\in\mathcal{S}}\big)\,\mathrm{diag}\,(x_1,\dots,x_n)\big((\mathrm{id})_{\mathfrak{p}\in\mathrm{Spec}(\mathcal{O}_K)}, \mathrm{diag}\,(r_1,\dots,r_n)\big)\big((h_\mathfrak{p})_{\mathfrak{p}\in\mathrm{Spec}(\mathcal{O}_K}$$

Now for $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_K)$, we have $u_\mathfrak{p}\,\mathrm{diag}\,(x_1,\dots,x_n)h_\mathfrak{p} = \mathrm{id}$, so both $u_\mathfrak{p}\,\mathrm{diag}\,(x_1,\dots,x_n)$ and $\mathrm{diag}\,\big(x_1^{-1},\dots,x_n^{-1}\big)u_\mathfrak{p}^{-1}$ lie in $\mathrm{GL}_n(\mathcal{O}_{K,\mathfrak{p}})$, which implies $x_i \in \mathcal{O}_{K,\mathfrak{p}}^\times$. This shows $x_i \in K^\times \cap \bigcap_{\mathfrak{p}\in\mathrm{Spec}(\mathcal{O}_K)}\mathcal{O}_{K,\mathfrak{p}}^\times = \mathcal{O}_K^\times$.

Now notice that in $\prod_{\nu\in\mathcal{S}} U_n(K_\nu)\backslash\mathrm{GL}_n(K_\nu)/O_\nu$, we have

$$\begin{aligned}\overline{\mathrm{diag}\,(r_1',\dots,r_n')} &= \overline{\mathrm{diag}\,(x_1,\dots,x_n)\,\mathrm{diag}\,(r_1,\dots,r_n)}\\ &= \overline{\mathrm{diag}(\varphi(x_i)r_i)_{i=1}^n},\end{aligned}$$

so from Lemma 6.4 and Lemma 6.5 we see $r_i' = \varphi(x_i)r_i$, which implies $\overline{r_i'} = \overline{r_i}$ in $\mathcal{B}$ for all $i \in [\![1,n]\!]$, hence shows that the original map is injective and finishes our proof. ☐

6.2 Intertwiner of the exponential function

Let us compute the result of applying the intertwiner to the function $\mathscr{C}(\lambda^*, s)(a)$. In particular, we claim the following.

Proposition 6.6. $M_w(\mathscr{C}(\lambda^*, s))(a) = \mathscr{C}(w^{-1}(\lambda^*), w^{-1}(s))(a) \cdot \Phi_{K,w^{-1}}(\lambda^*, s)$.

Let us show this. We will first deal with the case $n = 2, w = (12)$. To simplify the notations, we will compute

$$M_w(\mathscr{C}(\lambda^*, s) \otimes \mathscr{C}(\mu^*, t))(a, b).$$

To do this, we will first recall the construction of the inverse map of $\operatorname{diag}: \mathcal{B}^n \xrightarrow{\sim} U_n(\mathbb{A}_K)A_n(K)\backslash \mathrm{GL}_n(\mathbb{A}_K)/\widehat{\mathcal{K}}_n$. Let $\mathcal{S}$ denote the set of Archimedean places. Take any $\bar{g} = \overline{\left((g_\mathfrak{p})_{\mathfrak{p}\in\mathrm{Spec}(\mathcal{O}_K)}, (g_\nu)_{\nu\in\mathcal{S}}\right)} \in U_n(\mathbb{A}_K)A_n(K)\backslash \mathrm{GL}_n(\mathbb{A}_K)/\widehat{\mathcal{K}}_n$. We write $g_\mathfrak{p} = u_\mathfrak{p} a_\mathfrak{p} k_\mathfrak{p} \in U_n(K_\mathfrak{p})A_n(\{p^{\mathbb{Z}}\})\mathrm{GL}_n(\mathcal{O}_{K,\mathfrak{p}}), g_\nu = u_\nu a_\nu k_\nu \in U_n(K_\nu)A_n(\mathbb{R}_+)O_\nu$, where O_ν is the maximal compact subgroup of K_ν, and $(p) = \mathfrak{p}$. Then g corresponds to

$$\left(\prod_{\mathfrak{p}\in\mathrm{Spec}(\mathcal{O}_K)} a_\mathfrak{p}^{-1}\right)\overline{(a_\nu)}_{\nu\in\mathcal{S}} = \overline{\left(a_\nu \prod_{\mathfrak{p}\in\mathbb{P}} |\nu(a_\mathfrak{p})|^{-1}\right)_{\nu\in\mathcal{S}}}$$

in $\mathcal{B}^n$ (after dediagonalizing). From this and the fact that $\mathscr{C}(\lambda^*, s)$ is a group homomorphism we see we can compute the integration on each place separately, then multiply them together to get the integration result. We split the computation into 3 parts.

Part 1: $\nu \in \mathcal{S}_\mathbb{R}$ (the set of real archimedean places). We want to find a explicit inverse map of $\operatorname{diag}: \mathbb{R}_+^2 \to U_2(\mathbb{R})\backslash \mathrm{GL}_2(\mathbb{R})/O_2$. To do this, pick $\overline{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} \in U_2(\mathbb{R})\backslash\mathrm{GL}_2(\mathbb{R})/O_2$, then we first do map $A \mapsto AA^T$ to map it to a symmetric form $\left(\begin{smallmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{smallmatrix}\right)$, and then note that in symmetric form $\left(\begin{smallmatrix} x & y \\ y & z \end{smallmatrix}\right)$, after letting $v_2' = v_2 - (y/x)v_1$, the symmetric form becomes $\left(\begin{smallmatrix} x & 0 \\ 0 & z-y^2/x \end{smallmatrix}\right)$. Thus, under this operation, the inverse map sends $\overline{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} \in U_2(\mathbb{R})\backslash\mathrm{GL}_2(\mathbb{R})/O_2$ to

$$\left(\sqrt{a^2+b^2}, \sqrt{c^2+d^2 - \frac{(ac+bd)^2}{a^2+b^2}}\right) = \left(\sqrt{a^2+b^2}, \frac{\left|\det\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right|}{\sqrt{a^2+b^2}}\right) \in \mathbb{R}_+^2.$$

Now let us start integrating. We have $U_w(\mathbb{R}) = \{\mathrm{id}\}$ and $U_w(\mathbb{R}) \setminus U(\mathbb{R}) = \left\{\left(\begin{smallmatrix} 1 & 0 \\ * & 1 \end{smallmatrix}\right)\right\}$, and apply the integral to (a, b) the place ν gives contribution

$$\begin{aligned}
&a_\nu^{1/2} b_\nu^{-1/2} \int_{x\in\mathbb{R}} \left(\delta^{1/2}\cdot\mathscr{C}(\lambda^*, s)\otimes\mathscr{C}(\mu^*, t)\right)\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\begin{pmatrix} a_\nu & 0 \\ 0 & b_\nu \end{pmatrix}\right)\mathrm{d}x \\
=\;& a_\nu^{1/2} b_\nu^{-1/2} \int_{x\in\mathbb{R}} \left(\delta^{1/2}\cdot\mathscr{C}(\lambda^*, s)\otimes\mathscr{C}(\mu^*, t)\right)\begin{pmatrix} a_\nu x & b_\nu \\ a_\nu & 0 \end{pmatrix}\mathrm{d}x \\
=\;& a_\nu^{1/2} b_\nu^{-1/2} \int_{x\in\mathbb{R}} \left(\sqrt{a_\nu^2 x^2 + b_\nu^2}\right)^{-1/2}\left(\frac{a_\nu b_\nu}{\sqrt{a_\nu^2x^2+b_\nu^2}}\right)^{1/2}(\mathscr{C}(\lambda^*, s)\otimes\mathscr{C}(\mu^*, t))\left(\sqrt{a_\nu^2x^2+b_\nu^2}, \frac{a_\nu b_\nu}{\sqrt{a_\nu^2x^2+b_\nu^2}}\right)\mathrm{d}x \\
=\;& a_\nu^{1/2} b_\nu^{-1/2} \int_{x=-\infty}^{\infty} (\mathscr{C}(\lambda^*, s-1/2)\otimes\mathscr{C}(\mu^*, t+1/2))\left(\sqrt{a_\nu^2x^2+b_\nu^2}, \frac{a_\nu b_\nu}{\sqrt{a_\nu^2x^2+b_\nu^2}}\right)\mathrm{d}x \\
=\;& a_\nu^{1/2} b_\nu^{-1/2} a_\nu^{t+1/2+2\pi i\mu_\nu^*} b_\nu^{t+1/2+2\pi i\mu_\nu^*} \int_{x=-\infty}^{\infty} \left(a_\nu^2x^2+b_\nu^2\right)^{\frac{s-t-1+2\pi i(\lambda_\nu^*-\mu_\nu^*)}{2}}\mathrm{d}x \\
=\;& a_\nu^{t+1+2\pi i\mu_\nu^*} b_\nu^{t+2\pi i\mu_\nu^*} b_\nu^{s-t-1+2\pi i(\lambda_\nu^*-\mu_\nu^*)}\frac{b_\nu}{a_\nu}\sqrt{\pi}\frac{\Gamma\left(\frac{t-s}{2}+\pi i(\mu_\nu^*-\lambda_\nu^*)\right)}{\Gamma\left(\frac{t-s+1}{2}+\pi i(\mu_\nu^*-\lambda_\nu^*)\right)} \\
=\;& a_\nu^{t+2\pi i\mu_\nu^*} b_\nu^{s+2\pi i\lambda_\nu^*}\sqrt{\pi}\frac{\Gamma\left(\frac{t-s}{2}+\pi i(\mu_\nu^*-\lambda_\nu^*)\right)}{\Gamma\left(\frac{t-s+1}{2}+\pi i(\mu_\nu^*-\lambda_\nu^*)\right)}
\end{aligned}$$

when $\mathrm{Re}(t-s) > 0$ since

$$\int_{x\in\mathbb{R}} \left(ax^2+b\right)^s\mathrm{d}x = b^s\sqrt{\frac{b}{a}}\sqrt{\pi}\frac{\Gamma(-s-1/2)}{\Gamma(-s)}$$

when $a, b \in \mathbb{R}_+, \operatorname{Re} s < -1/2$.

Part 2: $\nu \in \mathcal{S}_\mathbb{C}$ (the set of complex archimedean places). Let us first get the inverse map of $\operatorname{diag}: \mathbb{R}_+^2 \xrightarrow{\sim} U_2(\mathbb{C})\backslash\mathrm{GL}_2(\mathbb{C})/\mathscr{U}_2$, where $\mathscr{U}_2$ is the unitary group. Take any $\overline{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} \in U_2(\mathbb{C})\backslash\mathrm{GL}_2(\mathbb{C})/\mathscr{U}_2$. Then the map $A \mapsto AA^*$ sends it to $\left(\begin{smallmatrix} |a|^2+|b|^2 & a\bar{c}+b\bar{d} \\ \bar{a}c+\bar{b}d & |c|^2+|d|^2 \end{smallmatrix}\right)$, and note that in Hermitian form $\left(\begin{smallmatrix} x & y \\ \bar{y} & z \end{smallmatrix}\right)$, after letting $v_2' = v_2 - (\bar{y}/x)v_1$, we see the matrix becomes $\left(\begin{smallmatrix} x & 0 \\ 0 & z-|y|^2/x \end{smallmatrix}\right)$. Thus, under this operation, the inverse map sends $\overline{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} \in U_2(\mathbb{C})\backslash \mathrm{GL}_2(\mathbb{C})/\mathscr{U}_2$ to

$$\left(\sqrt{|a|^2+|b|^2}, \sqrt{|c|^2+|d|^2-\frac{\left|a\bar{c}+b\bar{d}\right|^2}{|a|^2+|b|^2}}\right) = \left(\sqrt{|a|^2+|b|^2}, \frac{\left|\det\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right|}{\sqrt{|a|^2+|b|^2}}\right) \in \mathbb{R}_+^2.$$

Then similarly, apply the place ν to (a,b) yields contribution

$$\begin{aligned}
&a_\nu b_\nu^{-1} \int_{x\in\mathbb{C}} \left(\delta^{1/2}\cdot \mathscr{C}(\lambda^*, s)\otimes \mathscr{C}(\mu^*, t)\right)\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\begin{pmatrix} a_\nu & 0 \\ 0 & b_\nu \end{pmatrix}\right) \mathrm{d}x \\
&= a_\nu b_\nu^{-1} \int_{x\in\mathbb{C}} \left(\delta^{1/2}\cdot \mathscr{C}(\lambda^*, s)\otimes \mathscr{C}(\mu^*, t)\right)\begin{pmatrix} a_\nu x & b_\nu \\ a_\nu & 0 \end{pmatrix} \mathrm{d}x \\
&= a_\nu b_\nu^{-1} \int_{x\in\mathbb{C}} (\mathscr{C}(\lambda^*, s-1/2)\otimes \mathscr{C}(\mu^*, t+1/2))\left(\sqrt{a_\nu^2|x|^2+b_\nu^2}, \frac{a_\nu b_\nu}{\sqrt{a_\nu^2|x|^2+b_\nu^2}}\right) \mathrm{d}x \\
&= a_\nu b_\nu^{-1} a_\nu^{2t+1+2\pi i\mu_\nu^*} b_\nu^{2t+1+2\pi i\mu_\nu^*} \int_{x\in\mathbb{C}} \left(a_\nu^2|x|^2+b_\nu^2\right)^{s-t-1+\pi i(\lambda_\nu^*-\mu_\nu^*)} \mathrm{d}x \\
&= 2\pi a_\nu^{2t+2+2\pi i\mu_\nu^*} b_\nu^{2t+2\pi i\mu_\nu^*} \int_{r=0}^{\infty} r\left(a_\nu^2 r^2+b_\nu^2\right)^{s-t-1+\pi i(\lambda_\nu^*-\mu_\nu^*)} \mathrm{d}r \\
&= 2\pi a_\nu^{2t+2+2\pi i\mu_\nu^*} b_\nu^{2t+2\pi i\mu_\nu^*} \frac{b_\nu^{2s-2t+2\pi i(\lambda_\nu^*-\mu_\nu^*)}}{2a_\nu^2(t-s+\pi i(\mu_\nu^*-\lambda_\nu^*))} \\
&= a_\nu^{2t+2\pi i\mu_\nu^*} b_\nu^{2s+2\pi i\lambda_\nu^*} \pi \frac{1}{t-s+\pi i(\mu_\nu^*-\lambda_\nu^*)}
\end{aligned}$$

when $\mathrm{Re}(t-s)>0$. Also, note that we should multiply by a factor of 2 for each complex place that comes from the Tamagawa measure.

Part 3: finite place $\mathfrak{p}\in \mathrm{Spec}(\mathcal{O}_K)$, where $\mathfrak{p}=(p)\subset \mathcal{O}_K$ is a prime ideal. First let us find out the A in $\left(\begin{smallmatrix} cp^i & 1 \\ 1 & 0 \end{smallmatrix}\right) = uAk \in U_2(K_\mathfrak{p})A_n(\{p^\mathbb{Z}\})\mathrm{GL}_2(\mathcal{O}_{K,\mathfrak{p}})$ for $i\in\mathbb{Z}, c\in \mathcal{O}_{K,\mathfrak{p}}^\times$. For $i\geqslant 0$ we can take $A=\mathrm{id}$. For $i<0$ we do column operation to make the matrix becomes $\left(\begin{smallmatrix} cp^i & 0 \\ 1 & -c^{-1}p^{-i} \end{smallmatrix}\right)$, then we see $A=\left(\begin{smallmatrix} p^i & 0 \\ 0 & p^{-i} \end{smallmatrix}\right)$.

Now let us start integrating. Denote the natural embedding $K\to\prod_{\nu\in\mathcal{S}} K_\nu, x\mapsto(\nu(x))_{\nu\in\mathcal{S}}$ by ι. Apply the place $\mathfrak{p}$ to (a,b) yields contribution

$$
\begin{aligned}
&\int_{x\in K_{\mathfrak{p}}} \left(\delta^{1/2}\cdot \mathscr{C}(\lambda^*,s)\otimes\mathscr{C}(\mu^*,t)\right)\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&0\\x&1\end{pmatrix}\right)\mathrm{d}x\\
&=\int_{x\in K_{\mathfrak{p}}} \left(\mathscr{C}(\lambda^*,s-1/2)\otimes\mathscr{C}(\mu^*,t+1/2)\right)\begin{pmatrix}x&1\\1&0\end{pmatrix}\mathrm{d}x\\
&=\sum_{j\in\mathbb{Z}}\int_{x\in p^j\mathcal{O}_{K,\mathfrak{p}}^\times} \left(\mathscr{C}(\lambda^*,s-1/2)\otimes\mathscr{C}(\mu^*,t+1/2)\right)\begin{pmatrix}x&1\\1&0\end{pmatrix}\mathrm{d}x\\
&=1+\sum_{j<0}\mathcal{N}(p)^{-j}\left(1-\frac{1}{\mathcal{N}(p)}\right)\left(\mathscr{C}(\lambda^*,s-1/2)\otimes\mathscr{C}(\mu^*,t+1/2)\right)\begin{pmatrix}|\nu(p)|^{-j}&0\\0&|\nu(p)|^j\end{pmatrix}_{\nu\in\mathcal{S}}\\
&=1+\sum_{j<0}\mathcal{N}(p)^{-j}\left(1-\frac{1}{\mathcal{N}(p)}\right)\exp(j\cdot 2\pi i(\mu^*-\lambda^*)\cdot\log|\iota(p)|)\mathcal{N}(p)^{j(t-s+1)}\\
&=1+\left(1-\frac{1}{\mathcal{N}(p)}\right)\sum_{j>0}\frac{1}{|\iota(p)|^{j\cdot 2\pi i(\mu^*-\lambda^*)}\mathcal{N}(p)^{j(t-s)}}\\
&=1+\left(\frac{\mathcal{N}(p)-1}{\mathcal{N}(p)}\right)\frac{1}{|\iota(p)|^{2\pi i(\mu^*-\lambda^*)}\mathcal{N}(p)^{t-s}-1}\\
&=\frac{|\iota(p)|^{2\pi i(\mu^*-\lambda^*)}\mathcal{N}(p)^{t-s+1}-1}{|\iota(p)|^{2\pi i(\mu^*-\lambda^*)}\mathcal{N}(p)^{t-s+1}-\mathcal{N}(p)}
\end{aligned}
$$

when $\mathrm{Re}(t-s)>0$. Also, note that we should multiply by a factor of $\mathrm{disc}(\mathcal{O}_K)^{-1/2}$ at the end that comes from the Tamagawa measure.

Combine them together, after letting $s=s_2-s_1$, $\lambda^*=\lambda_2^*-\lambda_1^*$, we get

$$
\begin{aligned}
&M_w(\mathscr{C}(\lambda_1^*,s_1)\otimes\mathscr{C}(\lambda_2^*,s_2))(a_1,a_2)\\
&=(\mathscr{C}(\lambda_2^*,s_2)\otimes\mathscr{C}(\lambda_1^*,s_1))(a_1,a_2)\cdot\mathrm{disc}(\mathcal{O}_K)^{-1/2}\cdot\pi^{\frac{r_1}{2}}\cdot(2\pi)^{r_2}\\
&\quad\cdot\prod_{\nu\in\mathcal{S}_{\mathbb{R}}}\frac{\Gamma\left(\frac{s}{2}+\pi i\lambda_\nu^*\right)}{\Gamma\left(\frac{s+1}{2}+\pi i\lambda_\nu^*\right)}\cdot\prod_{\nu\in\mathcal{S}_{\mathbb{C}}}\frac{1}{s+\pi i\lambda_\nu^*}\\
&\quad\cdot\prod_{(p)\in\mathrm{Spec}(\mathcal{O}_K)}\frac{|\iota(p)|^{2\pi i\lambda^*}\mathcal{N}(p)^{s+1}-1}{|\iota(p)|^{2\pi i\lambda^*}\mathcal{N}(p)^{s+1}-\mathcal{N}(p)}
\end{aligned}
$$

when $\mathrm{Re}\,s>1$ (for each component we need $\mathrm{Re}\,s>0$, but for the product to converge, we need $\mathrm{Re}\,s>1$).

As a result, for $\mathrm{Re}(s_2-s_1)>1$, we see

$$M_w(\mathscr{C}(\lambda_1^*,s_1)\otimes\mathscr{C}(\lambda_2^*,s_2))(a_1,a_2)=(\mathscr{C}(\lambda_2^*,s_2)\otimes\mathscr{C}(\lambda_1^*,s_1))(a_1,a_2)\Phi_K(\lambda_2^*-\lambda_1^*,s_2-s_1).$$

To figure out $M_w\mathscr{C}(\lambda^*,s)$ in the general case, we need some extra works. For $w\in\mathfrak{S}_n$ we use $\mathfrak{I}_w$ to denote the set $\left\{(i,j)\in[\![1,n]\!]^2 : i<j, w(i)>w(j)\right\}$. First, let us prove a lemma.

Lemma 6.7. *$(a_{ij})_{i,j}\in U_w$ if and only if $a_{ij}=0$ and $a_{w^{-1}(i)w^{-1}(j)}=0$ when $i<j$; or equivalently, $(a_{ij})\in U$ and $a_{ij}=0$ for $(j,i)\in\mathfrak{I}_w$.*

Proof. Note that

$$
\begin{aligned}
(a_{ij})_{i,j} \in U_w &\iff (a_{ij})_{i,j} \in U \text{ and } w(a_{ij})_{i,j}w^{-1} \in U \\
&\iff \left\{(a_{ij})_{i,j}, (a_{w^{-1}(i)w^{-1}(j)})_{i,j}\right\} \subset U \\
&\iff a_{ij} = a_{w^{-1}(i)w^{-1}(j)} = 0 \text{ for } i < j
\end{aligned}
$$

as desired. □

Corollary 6.8. *If $t = (i\ j)$ is a transposition with $i < j$, then $U_t = \{(a_{ij})_{i,j} \in U : a_{ji} = 0\}$.*

Corollary 6.9. *If $t = (i\ j)$ is a transposition with $i < j$, then each equivalence class in $U_t\backslash U$ has a representative $(a_{i'j'})_{i',j'}$ with $a_{i'i'} = 1$ and $a_{i'j'} = 0$ for all $i' \neq j'$ except for possibly $(i', j') = (j, i)$.*

Proof. Let $b = (b_{k\ell}) \in U_t\backslash U$ and let $b^{-1} = (c_{k\ell})_{k,\ell}$ be its inverse. Let $d = (d_{k\ell})_{k,\ell} \in U_t$ be such that $d_{k\ell} = c_{k\ell}$ for $(k, \ell) \neq (i, j)$ and $d_{ij} = 0$. Note that $h = db$ is also a representative of $[b]$, but now $h_{kk} = 1$ and $h_{k\ell} = 0$ for all $k \neq \ell$ except for possibly $\ell = i$. Now, consider $m = (m_{ij}) \in U_t$, with $m_{kk} = 1$, $m_{ki} = -h_{ki}$ for $k \neq j$, and all the entries of m equal to 0. Note that, if $n = mh$, we have that $n_{kk} = 1$, $n_{k\ell} = \sum_r m_{kr}h_{r\ell} = 0$ if $k \neq \ell$ and $\ell \neq i$, and $n_{ki} = \sum_r m_{kr}h_{ri} = h_{ki} - h_{ki} = 0$ if $k \neq j$. That is, $n = mh = mdb$ is the desired representative of $[b]$ with $n_{kk} = 1$, $n_{k\ell} = 0$ if $k \neq \ell$ except for possibly $(k, \ell) \neq (j, i)$, as desired.

Proposition 6.10. *If $w \in \mathfrak{S}_n$ and t is a transposition of the form $(\ell\ \ell + 1)$ such that $\#\mathfrak{I}_{wt} > \#\mathfrak{I}_w$, then we have $M_t \circ M_w = M_{wt}$.*

Proof. For $w \in \mathfrak{S}_n$, we let $U^w = w^{-1}Uw$. It suffices to show that $\overline{M}_t \circ \overline{M}_w = \overline{M}_{wt}$. We have

$$
\begin{aligned}
\overline{M}_t\overline{M}_w\psi(g) &= \int_{v\in U_t\backslash U}\int_{u\in U_w\backslash U} \psi(wutvg)\,\mathrm{d}u\,\mathrm{d}v \\
&= \int_{v\in U_t\backslash U, u\in U_w\backslash U} \psi(wt(tut)vg)\,\mathrm{d}u\,\mathrm{d}v \\
&= \int_{u\in(U^{wt}\cap U^t)\backslash U^t, v\in(U^t\cap U)\backslash U} \psi(wtuvg)\,\mathrm{d}u\,\mathrm{d}v
\end{aligned}
$$

Now we introduce a lemma.

Lemma 6.11. *For transposition $t = (\ell\ \ell + 1)$ and $w \in \mathfrak{S}_n$ such that $\#\mathfrak{I}_{wt} > \#\mathfrak{I}_w$, the natural map between sets*

$$
(U^{wt} \cap U) \setminus (U^t \cap U) \to (U^{wt} \cap U^t) \setminus U^t
$$

is a bijection (note that $U^{wt} \cap U \subset U^t$).

Proof. We first show that it is injective. Take any $g, g' \in U^t \cap U$ such that there exists $h \in U^{wt} \cap U^t$ with $hg = g'$. Then $h = g'g^{-1} \in U^t \cap U$ (note that both U and U^t are groups). Thus, $\bar{g} = \overline{g'}$ in $(U^{wt} \cap U) \setminus (U^t \cap U)$ and so the map is injective.

Now we prove the surjectivity. For $x \in \mathbb{A}_K$, let $A(x)$ be the matrix with $A(x)_{ii} = 1$ for $i \in [\![1, n]\!]$ and $A(x)_{\ell,\ell+1} = x, A(x)_{ij} = 0$ for all other ij's. Then we claim that $A(x) \in U^{wt}$ for any $x \in \mathbb{A}_K$. This is because the condition $\#\mathfrak{I}_{wt} > \#\mathfrak{I}_w$ shows $w(\ell) < w(\ell + 1)$, and U^{wt} consists of matrices A with $A_{(wt)^{-1}(i),(wt)^{-1}(j)} = 0$ for $i < j$. Thus, $A(x) \notin U^{wt}$ only when there exists $i < j$ such that $(wt)^{-1}(i) = \ell, (wt)^{-1}(j) = \ell + 1$, i.e., $i = wt(\ell) = w(\ell + 1), j = wt(\ell + 1) = w(\ell)$. However, then the assumption $w(\ell) < w(\ell + 1)$ gives a contradiction.

Now given any $\bar{g} \in (U^{wt} \cap U^t) \setminus U^t$, we have $\bar{g} = \overline{Y(-g_{\ell,\ell+1})g}$, and the latter is in the image of the map, so we are done. □

Thus, we see

$$
\overline{M}_t\overline{M}_w\psi(g) = \int_{u\in(U^{wt}\cap U)\backslash(U^t\cap U), v\in(U^t\cap U)\backslash U} \psi(wtuvg)\,\mathrm{d}u\,\mathrm{d}v.
$$

Therefore, it suffices to show that

$$\int_{u\in(U^{wt}\cap U)\setminus(U^t\cap U),v\in(U^t\cap U)\setminus U}\psi(wtuvg)\,\mathrm{d}u\,\mathrm{d}v=\int_{x\in(U^{wt}\cap U)\setminus U}\psi(wtxg)\,\mathrm{d}x.$$

Note that from Corollary 6.9 we see there is a natural bijection between sets

$$Y\to(U^t\cap U)\setminus U$$

where $Y\subset U$ is the subset $\{g\in U:g_{ij}=0\text{ for }i\neq j\text{ except }(i,j)=(\ell+1,\ell)\}$.

Lemma 6.12. *There is a bijection between sets*

$$(U^{wt}\cap U)\setminus(U^t\cap U)\times(U^t\cap U)\setminus U\to(U^{wt}\cap U)\setminus(U^t\cap U)\times Y\xrightarrow{\cdot}(U^{wt}\cap U)\setminus U,$$

where $\cdot$ denote the usual product.

Proof. It suffices to show that the $\cdot$ part is a bijection. We first show the injectivity. Let $Y(x)=A(x)^T$, where $A(x)$ is the matrix defined in Lemma 6.11. Take any $g,g'\in U^t\cap U,x,x'\in\mathbb{A}_K,h\in U^{wt}\cap U$ such that $hgY(x)=g'Y(x')$. Then by looking at the $(\ell+1,\ell)$ entry we see $x=x'$, so $hg=g'$ and $\bar g=\overline{g'}$ in $\overline{(U^{wt}\cap U)\setminus(U^t\cap U)}$, as desired.

Now we show the surjectivity. Take any $\bar g\in(U^{wt}\cap U)\setminus U$, then we see it is the image of $(\overline{gY(-g_{\ell+1,\ell})},Y(g_{\ell+1,\ell}))$, so we are done. ☐

We now let $G=(U^{wt}\cap U)\setminus U$ and $H=(U^{wt}\cap U)\setminus(U^t\cap U)$, then Lemma 6.12 shows Y is a fundamental domain for $H\setminus G$. From knowledge about Haar measure (for example, see [OV99]) we see for any $f\in C_c(G)$, we have

$$\int_Y\int_H f(hy)\,\mathrm{d}\mu_H\,\mathrm{d}\mu_{H\setminus G}=c\int_G f(g)\,\mathrm{d}\mu_G$$

for some constant $c\in\mathbb{R}$. We now take f to be any measurable function $G\to\{0,1\}$, then after taking any fundamental region for H we see $c=1$ by Cavalieri's principle, which finishes the proof. ☐

Corollary 6.13. *For $w,w'\in\mathfrak{S}_n$ with $\#\mathfrak{I}_{ww'}=\#\mathfrak{I}_w+\#\mathfrak{I}_{w'}$, we have $M_{w'}M_w=M_{ww'}$.*

Proof. We use induction on $\#\mathfrak{I}_{w'}$. When $\#\mathfrak{I}_{w'}=1$, the statement follows from Proposition 6.10, so we assume $\#\mathfrak{I}_{w'}>1$. Then we write $w'=tw_0$, where t is a transposition of adjacent elements and $\#\mathfrak{I}_{w_0}=\#\mathfrak{I}_{w'}-1$. Then we have $\#\mathfrak{I}_{wt}=\#\mathfrak{I}_w+1$. Now from Proposition 6.10 and induction hypothesis we see

$$\begin{aligned}M_{w'}M_w&=M_{w_0}M_tM_w\\&=M_{w_0}M_{wt}\\&=M_{wtw_0}\\&=M_{ww'},\end{aligned}$$

as desired. ☐

Lemma 6.14. *For $w,w'\in\mathfrak{S}_n,\lambda^*\in(\Lambda^*)^n,z\in(\Lambda^*\times\mathbb{C})^n$, we have*

$$\Phi_{K,w'}(w(z))\Phi_{K,w}(z)=\Phi_{K,w'w}(z).$$

Proof. We first show this for $w'=t=(\ell\ \ell+1)$ being a transposition of adjacent elements. To do this, we split into 2 cases.

Case 1: $\#\mathfrak{I}_{tw}>\#\mathfrak{I}_w$. Then we have

$$\begin{aligned}\Phi_{K,w'}(w(z)))\Phi_{K,w}(z)&=\Phi_K(z_{w^{-1}(\ell+1)}-z_{w^{-1}(\ell)})\Phi_{K,w}(z)\\&=\Phi_{K,w'w}(z)\end{aligned}$$

because $w^{-1}(\ell+1) > w^{-1}(\ell)$ from assumption that $\#\mathfrak{I}_{tw} > \#\mathfrak{I}_w$, and it is the only different of $\Phi_{K,w}$ and $\Phi_{K,tw}$ since no extra difference is allowed (we have $\#\mathfrak{I}_{tw} = \#\mathfrak{I}_w + 1$).

Case 2: $\#\mathfrak{I}_{tw} < \#\mathfrak{I}_w$. Then we have $w = tw_0$ for some w_0 that satisfies $\#\mathfrak{I}_{w_0} < \#\mathfrak{I}_w$. As a result, we see

$$\begin{aligned}\Phi_{K,w'}(w(z))\Phi_{K,w}(z) &= \Phi_K(z_{w_0^{-1}t(\ell+1)} - z_{w_0^{-1}t(\ell)})\Phi_{K,tw_0}(z)\\ &= \Phi_K(z_{w_0^{-1}(\ell)} - z_{w_0^{-1}(\ell+1)})\Phi_K(z_{w_0^{-1}(\ell+1)} - z_{w_0^{-1}(\ell)})\Phi_{K,w_0}(z)\\ &= \Phi_{K,tw}(z)\end{aligned}$$

because $\Phi_K(z)^{-1} = \Phi_K(-z)$.

Now we deal with the general case. We use induction on $\#\mathfrak{I}_{w'}$. When $\#\mathfrak{I}_{w'} = 1$, then w' is a transposition of adjacent elements and we are done from previous discussion, so we assume $\#\mathfrak{I}_{w'} > 1$. Then we can write $w' = w_0 t$ for some transposition of adjacent elements t and $\#\mathfrak{I}_{w_0} = \#\mathfrak{I}_{w'} - 1$. Then we have

$$\begin{aligned}\Phi_{K,w'}(w(z))\Phi_{K,w}(z) &= \Phi_{K,w_0t}(w(z))\Phi_{K,w}(z)\\ &= \Phi_{K,w_0}(tw(z))\Phi_{K,t}(w(z))\Phi_{K,w}(z)\\ &= \Phi_{K,w_0}(tw(z))\Phi_{K,tw}(z)\\ &= \Phi_{w_0tw}(z)\\ &= \Phi_{w'w}(z),\end{aligned}$$

from induction hypothesis and the case when w' is a transposition of adjacent elements, as desired. $\square$

Now for $w \in \mathfrak{S}_n$, let $\mathbb{C}^n_w = \{(s_1, \ldots, s_n) \in \mathbb{C}^n : \mathrm{Re}(s_j - s_i) > 1 \text{ for } (i,j) \in \mathfrak{I}_w\}$. We are at a position to establish the integration result for the general case.

Proposition 6.15. *Let $w \in \mathfrak{S}_n, s \in \mathbb{C}^n_{w^{-1}}, \lambda^* \in (\Lambda^*)^n$, then*

$$M_w\mathscr{C}(\lambda^*, s) = \mathscr{C}(w^{-1}(\lambda^*), w^{-1}(s))\Phi_{K,w^{-1}}(\lambda^*, s).$$

Proof. We use induction on $\#\mathfrak{I}_w$. If $\#\mathfrak{I}_w = 1$, then w is a transposition of adjacent elements, so from previous discussion we see the equation follows. Thus, we assume $\#\mathfrak{I}_w > 1$. Then we can write $w = tw_0$ with t being a transposition of adjacent elements and $\#\mathfrak{I}_{w_0} = \#\mathfrak{I}_w - 1$. Then

$$\begin{aligned}M_w\mathscr{C}(\lambda^*, s) &= M_{w_0}M_t(\mathscr{C}(\lambda^*, s))\\ &= M_{w_0}\mathscr{C}(t(\lambda^*), t(s))\Phi_{K,t}(\lambda^*, s)\\ &= \mathscr{C}(w_0^{-1}t(\lambda^*), w_0^{-1}t(s))\Phi_{K,w_0^{-1}}(t(\lambda^*), t(s))\Phi_{K,t}(\lambda^*, s)\\ &= \mathscr{C}(w^{-1}(\lambda^*), w^{-1}(s))\Phi_{K,w_0^{-1}t}(\lambda^*, s)\\ &= \mathscr{C}(w^{-1}(\lambda^*), w^{-1}(s))\Phi_{K,w^{-1}}(\lambda^*, s),\end{aligned}$$

from Proposition 6.10, the case when w is a transposition of adjacent elements, the induction hypothesis, and Lemma 6.14, as desired. $\square$

6.3 Intertwiner and the Fourier transform

Let us compute the result of composing the intertwiner and the Fourier transform. More explicitly, we claim the following.

Proposition 6.16. $\mathcal{F}(M_w(\varphi))(\lambda^*, s) = \mathcal{F}(\varphi)(w(\lambda^*), w(s)) \cdot \Phi_{K,w}(\lambda^*, s)$

Proof. Choose F such that $\varphi = \mathcal{G}F$. By 4.1, we have that

$$
\begin{aligned}
\mathcal{F}(M_w(\varphi))(\lambda^*,s) &= \int_{a\in D}\int_{u\in U_w(\mathbb{A}_K)\backslash U(\mathbb{A}_K)} (\mathcal{G}F)(wua)\,\mathrm{d}u\mathscr{C}(\lambda^*,s)(a)\,\mathrm{d}^*a\\
&= \int_{a\in D}\int_{u\in U_w(\mathbb{A}_K)\backslash U(\mathbb{A}_K)} \frac{1}{(2\pi i)^{n(r_1+2r_2)}}\sum_{\eta^*\in\Lambda^*}\int_{t\in\sigma_0+i\mathbb{R}^n} F(\eta^*,t)\mathscr{C}(-\lambda^*,-t)(wua)\,\mathrm{d}t\,\mathrm{d}u\mathscr{C}(\lambda^*,s)(a)\,\mathrm{d}^*a\\
&= \int_{a\in D}\frac{1}{(2\pi i)^{n(r_1+2r_2)}}\sum_{\eta^*\in\Lambda^*}\int_{t\in\sigma_0+i\mathbb{R}^n}\int_{u\in U_w(\mathbb{A}_K)\backslash U(\mathbb{A}_K)} F(\eta^*,t)\mathscr{C}(-\lambda^*,-t)(wua)\,\mathrm{d}u\,\mathrm{d}t\mathscr{C}(\lambda^*,s)(a)\,\mathrm{d}^*a\\
&= \int_{a\in D}\frac{1}{(2\pi i)^{n(r_1+2r_2)}}\sum_{\eta^*\in\Lambda^*}\int_{t\in\sigma_0+i\mathbb{R}^n} F(\eta^*,t)M_w(\mathscr{C}(-\lambda^*,-t))(a)\mathscr{C}(\lambda^*,s)(a)\,\mathrm{d}^*a\\
&= \int_{a\in D}\frac{1}{(2\pi i)^{n(r_1+2r_2)}}\sum_{\eta^*\in\Lambda^*}\int_{t\in\sigma_0+i\mathbb{R}^n} F(\eta^*,t)\mathscr{C}(-w^{-1}(\lambda^*),-w^{-1}(t))(a)\Phi_{K,w^{-1}}(-\lambda^*,-t)\,\mathrm{d}t\mathscr{C}(\lambda^*,s)(a)\,\mathrm{d}^*a\\
&= \int_{a\in D}\mathcal{G}(F\cdot(\Phi_{K,w^{-1}}\circ(-1)))(w(a))\mathscr{C}(\lambda^*,s)(a)\,\mathrm{d}^*a\\
&= \int_{a\in D}\mathcal{G}(F\cdot(\Phi_{w^{-1}}\circ(-1)))(a)\mathscr{C}(w(\lambda^*),w(s))(a)\,\mathrm{d}^*a\\
&= \mathcal{F}(\mathcal{G}(F\cdot(\Phi_{K,w^{-1}}\circ(-1))))(w(\lambda^*),w(s))\\
&= F(w(\lambda^*),w(s))\cdot\Phi_{K,w^{-1}}(-w(\lambda^*),-w(s))\\
&= \mathcal{F}(\varphi)(w(\lambda^*),w(s))\cdot\Phi_{K,w}(\lambda^*,s)
\end{aligned}
$$

as desired. □

7 Isomorphism of the Hall algebra and the shuffle algebra

Let us show that the spherical Hall algebra of $\overline{\mathrm{Spec}(\mathcal{O}_K)}$ and Paley-Wiener the shuffle algebra associated to $\Phi_K(s)$ are isomorphic. First, we define the constant term CT_n and its twist $\widetilde{\mathrm{CT}}_n$. We will show important properties these constant term operators satisfy. Afterwards, we define the operator $\mathrm{Ch}_n = \mathcal{F}\circ\widetilde{\mathrm{CT}}_n$, and we will show that $\mathrm{Ch} = \bigoplus_{n\geqslant 0}\mathrm{Ch}_n$ is the desired isomorphism between SH and $\mathcal{SH}(\Phi_K)_{\mathcal{PW}}$.

7.1 The constant term

Let B_n be the lower triangular Borel subgroup of GL_n, and let U_n be the unipotent radical of B_n. We define the constant term CT_n as an operator $\mathrm{CT}_n : H_n \to C_c^\infty(\mathcal{B}^n)$ given by

$$
\mathrm{CT}_n(a_1,\dots,a_n) = \int_{u\in U_n(\mathcal{O}_K)\backslash U_n(R)} f(u\cdot\mathrm{diag}(a_1,\dots,a_n))\,\mathrm{d}u = \int_{u_{\mathbb{A}}\in U_n(K)\backslash U_n(\mathbb{A}_K)} f(u_{\mathbb{A}}\cdot\mathrm{diag}(a_1,\dots,a_n))\,\mathrm{d}u_{\mathbb{A}}
$$

Here, du and $du_{\mathbb{A}}$ are the normalized Hall measures on $U_n(R)$ and $U_n(\mathbb{A}_K)$, respectively, such that $U_n(\mathcal{O}_K)$ and $U_n(K)$ have volume 1.

Proposition 7.1. *For any $f\in H_n$, there is some $c\in\mathbb{R}_+$ such that $\mathrm{Supp}(\mathrm{CT}_n(f))$ is contained in the domain*

$$
\left\{(a_1,\dots,a_n)\in\mathcal{B}^n : |a_1|_{\mathcal{B}}\leqslant c, |a_1a_2|_{\mathcal{B}}\leqslant c,\dots,|a_1\dots a_{n-1}|_{\mathcal{B}}\leqslant c, \frac{1}{c}\leqslant|a_1\dots a_n|_{\mathcal{B}}\leqslant c\right\}.
$$

Proof. The proof is the same as in [KSV12]. □

We use the Haar measures $\mathrm{d}g = \frac{\prod_{i,j}^n \mathrm{d}g_{ij}}{\det(g)^n}$ and $\mathrm{d}^*a = \prod_{i=1}^n \frac{\mathrm{d}a_i}{a_i}$ for $\mathrm{GL}_n(R)$ and $\mathcal{B}^n$, respectively. Note that the

Iwasawa decomposition of $\mathrm{GL}_n(R)$ yields

$$\mathrm{GL}_n(R) = U_n(R) \cdot \mathcal{B}^n \cdot \mathcal{K}_n$$

so we will use the notation $g = u \cdot a \cdot k$ for an element g of $\mathrm{GL}_n(R)$. If $\mathrm{d}u$ and $\mathrm{d}k$ are the Haar measures on $U_n(R)$ and $\mathcal{K}_n$, we have that

$$\mathrm{d}g = \delta(a)\,\mathrm{d}u \cdot \mathrm{d}k \cdot \mathrm{d}a$$

where

$$\delta(a) = \delta(g) = \prod_{1 \leqslant i < j \leqslant n} \frac{a_j}{a_i}$$

is the Iwasawa Jacobian.

Let us define the following positive definite Hermitian scalar products for H_n and $C_c^\infty(\mathcal{B}^n)$, respectively:

$$\langle f_1, f_2 \rangle_H = \int_{g \in \mathrm{GL}_n(\mathcal{O}_K) \backslash \mathrm{GL}_n(R)} f_1(g) \overline{f_2(g)}\,\mathrm{d}g \quad \text{and} \quad \langle \varphi_1, \varphi_2 \rangle = \frac{1}{2^n} \int_{a \in \mathcal{B}^n} \varphi_1(a) \overline{\varphi_2(a)}\,\mathrm{d}^* a$$

We define the *twisted constant term* $\widetilde{\mathrm{CT}}_n$ as

$$\widetilde{\mathrm{CT}}_n(f)(a_1, \ldots, a_n) = \mathrm{CT}_n(f)(a_1, \ldots, a_n) \cdot \delta(a)^{\frac{1}{2}}$$

Proposition 7.2. *The map $\widetilde{\mathrm{CT}}_n : H_n \to C^\infty(\mathcal{B}^n)$ is adjoint to $*_{1^n} : H_1^{\otimes n} \to H_n$, that is, we have*

$$\langle *_{1^n}(\varphi), f \rangle_H = \left\langle \varphi, \widetilde{\mathrm{CT}}_n(f) \right\rangle$$

Proof. By Proposition 6.1, we have that

$$\begin{aligned}
\langle *_{1^n}(\varphi), f \rangle_H &= \int_{g \in \mathrm{GL}_n(\mathcal{O}_K) \backslash \mathrm{GL}_n(R)} \overline{f(g)} \sum_{\gamma \in B_n(\mathcal{O}_K) \backslash \mathrm{GL}_n(\mathcal{O}_K)} \widetilde{\varphi}(\gamma g)\,\mathrm{d}g \\
&= \int_{x \in B_n(\mathcal{O}_K) \backslash \mathrm{GL}_n(R)} \overline{f(x)} \widetilde{\varphi}(x)\,\mathrm{d}x \\
&= \int_{x \in B_n(\mathcal{O}_K) \backslash \mathrm{GL}_n(R)} \overline{f(x)} \varphi(x) \delta(x)^{-\frac{1}{2}}\,\mathrm{d}x \\
&= \frac{1}{2^n} \int_{y \in U_n(\mathcal{O}_K) \backslash \mathrm{GL}_n(R)} \overline{f(y)} \varphi(y) \delta(y)^{-\frac{1}{2}}\,\mathrm{d}y \\
&= \frac{1}{2^n} \int_{z \in U(R) \backslash \mathrm{GL}_n(R)} \int_{u \in U_n(\mathcal{O}_K) \backslash U_n(R)} \overline{f(uz)} \varphi(z) \delta(z)^{-\frac{1}{2}}\,\mathrm{d}u\,\mathrm{d}z \\
&= \frac{1}{2^n} \int_{a \in \mathcal{B}^n} \overline{\mathrm{CT}_n(f)(a)} \varphi(a) \delta(a)^{\frac{1}{2}}\,\mathrm{d}^* a \\
&= \left\langle \varphi, \widetilde{\mathrm{CT}}_n(f) \right\rangle
\end{aligned}$$

as desired. □

Proposition 7.3. *The map $\widetilde{\mathrm{CT}}_n : SH_n \to C^\infty(\mathcal{B}^n)$ is injective.*

Proof. By definition of SH, an element $f \in SH_n$ has the form $f = *_{1^n}(\varphi)$ for some $\varphi \in H_1^{\otimes n} \subset C_c^\infty(\mathcal{B}^n)$. Now, assume that $f \neq 0$, which implies $\varphi \neq 0$. We will show that $\widetilde{\mathrm{CT}}_n(f) \neq 0$, from which injectivity will follow. Indeed, note that, as by Proposition 7.2 we have that

$$\left\langle \varphi, \widetilde{\mathrm{CT}}_n(f) \right\rangle = \left\langle \varphi, \widetilde{\mathrm{CT}}_n(*_{1^n}(\varphi)) \right\rangle = \langle *_{1^n}(\varphi), *_{1^n}(\varphi) \rangle = \langle f, f \rangle_H > 0.$$

Then, in particular, $\widetilde{\mathrm{CT}}_n(f) \neq 0$, as desired. $\square$

Proposition 7.4. *We have that*

$$\widetilde{\mathrm{CT}}_{n'+n''}(f' * f'') = \sum_{w \in \mathrm{Sh}(n',n'')} M_{w^{-1}}(\widetilde{\mathrm{CT}}_{n'}(f') \otimes \widetilde{\mathrm{CT}}_{n''}(f''))$$

Proof. Recall that the Grassmannian $\mathrm{Gr}_{n'}(K^n) = P_{n',n''}(K)\backslash \mathrm{GL}_n(K)$ splits under the right $U(K)$-action, into $\binom{n}{n'}$ orbits (the Schubert cells)

$$\Sigma_w = P_{n',n''}(K)\backslash w^{-1}U(K)$$

where $w \in \mathrm{Sh}(n', n'')$. Now, by this and Proposition 2.3, notice that

$$\begin{aligned}
\widetilde{\mathrm{CT}}_n(f' * f'') &= \int_{u\in U(K)\backslash U(\mathbb{A}_K)} \sum_{\gamma \in B_{n',n''}(K)\backslash \mathrm{GL}_n(K)} f(\gamma u g)\delta(g)^{\frac{1}{2}}\, \mathrm{d}u \\
&= \sum_{w\in \mathrm{Sh}(n',n'')} \int_{u\in U(K)\backslash U(\mathbb{A}_K)} \sum_{v \in U_{w^{-1}}(K)\backslash U(K)} f(w^{-1}vug)\delta(g)^{\frac{1}{2}}\, \mathrm{d}u \\
&= \sum_{w \in Sh(n',n'')} M_{w^{-1}}(\widetilde{\mathrm{CT}}_{n'}(f') \otimes \widetilde{\mathrm{CT}}_{n''}(f''))
\end{aligned}$$

as desired. $\square$

Proposition 7.5. *Let $\varphi_1, \dots, \varphi_n \in C_c^\infty(\mathcal{B})$ and $\varphi \in \varphi_1 \otimes \cdots \otimes \varphi_n \in C_c^\infty(\mathcal{B}^n)$. Then*

$$\widetilde{\mathrm{CT}}_n(*_{1^n}(\varphi)) = \sum_{w\in\mathfrak{S}_n} M_w(\varphi).$$

Proof. The proof is similar to the previous one. Note that, by Proposition 6.1,

$$\begin{aligned}
\widetilde{\mathrm{CT}}_n(*_{1^n}(\varphi)) &= \int_{u\in U(K)\backslash U(\mathbb{A}_K)} \sum_{\gamma\in B_n(\mathcal{O}_K)\backslash \mathrm{GL}_n(\mathcal{O}_K)} \widetilde{\varphi}(\gamma u g)\delta(g)^{\frac{1}{2}}\, \mathrm{d}u \\
&= \sum_{w\in\mathfrak{S}_n} M_w(\varphi)
\end{aligned}$$

as desired. $\square$

7.2 The Ch homomorphism

We define the operator $\mathrm{Ch}_n : H_1^{\otimes n} \to \mathcal{PW}((\Lambda^* \times \mathbb{C})^n)$ as $\mathrm{Ch}_n(f) = \mathcal{F}(\widetilde{\mathrm{CT}}_n(f))$ for $n \geqslant 1$ and $\mathrm{Ch}_0 : \mathbb{C} \to \mathbb{C}$ as $\mathrm{Ch}_0(c) = c$ for $n = 0$. Let $\mathrm{Ch} : SH \to \mathcal{SH}(\Phi_K)_{\mathcal{PW}}$ be defined by $\mathrm{Ch} = \bigoplus_{n\geqslant 0} \mathrm{Ch}_n$.

Proposition 7.6. *The map* Ch *is well-defined as a map of sets.*

Proof. It suffices to show that $\mathrm{Ch}_n(f)$ is well-defined for any f. We show that $\mathrm{Ch}_n(f)$ is well-defined and meromorphic in $(\Lambda^*)^n \times \mathbb{C}_{>0}$, and then we can extend it to a meromorphic function in all of $(\Lambda^* \times \mathbb{C})^n$. The argument is the same as in [KSV12]. Finally, as $\mathcal{F}(H_1) = \mathcal{PW}(\Lambda^* \times \mathbb{C})$, $\mathcal{F}(SH) \subset \mathcal{SH}(\Phi_K)_{\mathcal{PW}}$, so Ch is well-defined, as desired. $\square$

Now that we have defined the map $\mathrm{Ch} : SH \to \bigoplus_{n\geqslant 0} \mathcal{PW}((\Lambda^* \times \mathbb{C})^n)$, we claim that this map is the desired isomorphism $SH \xrightarrow{\sim} \mathcal{SH}(\Phi_K)_{\mathcal{PW}}$.

Proposition 7.7. *The map* $\mathrm{Ch}(f)=0$ *if and only if* $f=0$.

Proof. Clearly $\mathrm{Ch}(0)=0$. Let us show the opposite. If $\mathrm{Ch}(f)=0$, we have that $\mathrm{Ch}_n(f)=0$ for all n. This implies that $\mathcal{F}(\widetilde{\mathrm{CT}}_n(f))=0$, which becomes $\widetilde{\mathrm{CT}}_n(f)=0$ after applying $\mathcal{G}$, but this implies $f=0$ by Proposition 7.3, so the result follows. $\square$

Proposition 7.8. *The map* $\mathrm{Ch}_1 : SH_1 = H_1 \to \mathcal{PW}(\Lambda^* \times \mathbb{C})$ *is an isomorphism.*

Proof. Note that $\widetilde{\mathrm{CT}}_1(f)(a)=f(a)$, so $\mathrm{Ch}_1(f)=\mathcal{F}(f)$. However, recall that by Proposition 5.1, $\mathcal{F}$ is an isomorphism, so Ch_1 is also an isomorphism, as desired. $\square$

Now, assume that Ch was an algebra homomorphism. In that case, by Proposition 7.7, Ch would be injective. Furthermore, by Proposition 7.8,

$$\mathrm{Ch}(SH)=\bigoplus_{n\geqslant 0}\mathrm{Ch}(H_1^{\otimes n})=\bigoplus_{n\geqslant 0}\mathrm{Ch}(H_1)^{\otimes n}=\bigoplus_{n\geqslant 0}\mathcal{PW}(\Lambda^*\times\mathbb{C})^{\otimes n}=\mathcal{SH}(\Phi_K)_{\mathcal{PW}}$$

so Ch would also be surjective. Thus, Ch would be the desired isomorphism. Let us show that Ch is an algebra homomorphism. To do this, we will introduce two lemmas. We use $\mathrm{Sh}(m,n)^{-1}$ to denote the set $\{w\in\mathfrak{S}_{m+n} : w^{-1}\in\mathrm{Sh}(m,n)\}$.

Lemma 7.9. *We have a bijection between sets*

$$\begin{aligned}\mathfrak{S}_m\times\mathfrak{S}_n\times\mathrm{Sh}(m,n)^{-1}&\xrightarrow{\sim}\mathfrak{S}_{m+n}\\(w',w'',w)&\mapsto(w'\times w'')w,\end{aligned}$$

where $w'\times w''$ *is the permutation that permutes the first* m *elements using* w' *and the last* n *elements using* w''.

Proof. It suffices to show that the map

$$\begin{aligned}\varphi:\mathrm{Sh}(m,n)\times\mathfrak{S}_m\times\mathfrak{S}_n&\to\mathfrak{S}_{m+n}\\(w,w',w'')&\mapsto w(w'\times w'')\end{aligned}$$

is bijective, since then we arrive at the desired conclusion by the following commutative diagram:

$$\begin{tikzcd}\mathfrak{S}_m\times\mathfrak{S}_n\times\mathrm{Sh}(m,n)^{-1} \arrow[r] \arrow[d, "\sim"', "\cdot^{-1}"] & \mathfrak{S}_{m+n} \arrow[d, "\sim"', "\cdot^{-1}"] \\ \mathrm{Sh}(m,n)\times\mathfrak{S}_m\times\mathfrak{S}_n \arrow[r, "\varphi"] & \mathfrak{S}_{m+n}\end{tikzcd}$$

In order to show that φ is bijective, we will first show it is surjective and deduce from the fact that both sides have the same cardinality $(m+n)!$. To see that φ is surjective, take any $\sigma\in\mathfrak{S}_{m+n}$. Then there exists $\sigma_1\times\sigma_2\in\mathfrak{S}_m\times\mathfrak{S}_n$ such that $\sigma(\sigma_1\times\sigma_2)(i)<w(\sigma_1\times\sigma_2)(j)$ for $(i,j)\in[\![1,m]\!]^2\sqcup[\![m+1,m+n]\!]^2$, i.e., $\sigma(\sigma_1\times\sigma_2)\in\mathrm{Sh}(m,n)$, so we see $\sigma=\varphi(\sigma(\sigma_1\times\sigma_2),\sigma_1^{-1},\sigma_2^{-1})\in\operatorname{im}\varphi$.

Now note that the domain of φ has cardinality $m!\cdot n!\cdot\binom{m+n}{m}=(m+n)!$, while the codomain of φ also has cardinality $(m+n)!$, so we see φ is injective and finishes our proof. $\square$

Lemma 7.10. *For* $w'\in\mathfrak{S}_m, w''\in\mathfrak{S}_n, w\in\mathrm{Sh}(m,n)^{-1}$, *we have*

$$\#\mathfrak{I}_{(w'\times w'')w}=\#\mathfrak{I}_{w'\times w''}+\#\mathfrak{I}_w.$$

More precisely, we have

$$\mathfrak{I}_{(w'\times w'')w}=\left(w^{-1}\mathfrak{I}_{w'\times w''}\right)\sqcup\mathfrak{I}_w.$$

Proof. The two maps

$$\begin{aligned}\mathfrak{I}_{w'\times w''} &\to \mathfrak{I}_{(w'\times w'')w}\\ (i,j) &\mapsto (w^{-1}(i), w^{-1}(j))\\ \mathfrak{I}_{w} &\to \mathfrak{I}_{(w'\times w'')w}\\ (i,j) &\mapsto (i,j)\end{aligned}$$

are injective, and the image of these two maps do not have intersection because otherwise there exist $i<j$ such that $(w^{-1}(i), w^{-1}(j)) \in \mathfrak{I}_w$, which implies $i>j$, a contradiction.

As a result, we see $\#\mathfrak{I}_{(w'\times w'')w} \geqslant \#\mathfrak{I}_{w'\times w''} + \#\mathfrak{I}_w$. However, for $w_1, w_2 \in \mathfrak{S}_{m+n}$, we have $\#\mathfrak{I}_{w_1 w_2} \leqslant \#\mathfrak{I}_{w_1} + \#\mathfrak{I}_{w_2}$, which shows $\#\mathfrak{I}_{(w'\times w'')w} = \#\mathfrak{I}_{w'\times w''} + \#\mathfrak{I}_w$, as desired. □

Corollary 7.11. *For $w'\times w'' \in \mathfrak{S}_m \times \mathfrak{S}_n, w \in \mathrm{Sh}(m,n)^{-1}$, we have*

$$M_w M_{w'\times w''} = M_{(w'\times w'')w}.$$

Proof. This follows from Corollary 6.13 and Lemma 7.10. □

As $\mathcal{F}$ and $\widetilde{\mathrm{CT}}_n$ are linear, $\mathrm{Ch}(f'+f'') = \mathrm{Ch}(f') + \mathrm{Ch}(f'')$ follows. We also have that $\mathrm{Ch}(1)=1$. It remains to show that $\mathrm{Ch}_n(f' * f'') = \mathrm{Ch}_{n'}(f') \circledS \mathrm{Ch}_{n''}(f'')$ for $f' \in H_1^{\otimes n'}$ and $f'' \in H_1^{\otimes n''}$. Let us do this. First, we compute $\mathrm{Ch}_n(f'*f'')(\lambda^*, s)$. If $f' = *_{1^{n'}}(\varphi')$ and $f'' = *_{1^{n''}}(\varphi'')$, by Proposition 7.4, Proposition 7.5, Corollary 7.11, Lemma 7.9, and Proposition 6.16,

$$\begin{aligned}\mathrm{Ch}_n(f'*f'')(\lambda^*, s) &= \mathcal{F}(\widetilde{\mathrm{CT}}_{n'+n''}(f'*f''))(\lambda^*, s)\\ &= \mathcal{F}\left(\sum_{w\in \mathrm{Sh}(n',n'')} M_{w^{-1}}(\widetilde{\mathrm{CT}}_{n'}(f') \otimes \widetilde{\mathrm{CT}}_{n''}(f''))\right)(\lambda^*, s)\\ &= \mathcal{F}\left(\sum_{w\in \mathrm{Sh}(n',n'')} M_{w^{-1}}\left(\sum_{w'\in\mathfrak{S}_{n'}} M_{w'}(\varphi') \otimes \sum_{w''\in\mathfrak{S}_{n''}} M_{w''}(\varphi'')\right)\right)(\lambda^*, s)\\ &= \mathcal{F}\left(\sum_{w\in \mathrm{Sh}(n',n'')} \sum_{w'\in\mathfrak{S}_{n'}} \sum_{w''\in\mathfrak{S}_{n''}} M_{w^{-1}}(M_{w'}(\varphi')) \otimes M_{w^{-1}}(M_{w''}(\varphi''))\right)(\lambda^*, s)\\ &= \mathcal{F}\left(\sum_{w\in \mathrm{Sh}(n',n'')} \sum_{\substack{w'\in\mathfrak{S}_{n'}\\ w''\in\mathfrak{S}_{n''}}} M_{(w'\times w'')w^{-1}}(\varphi)\right)(\lambda^*, s)\\ &= \mathcal{F}\left(\sum_{\sigma\in\mathfrak{S}_n} M_\sigma(\varphi)\right)(\lambda^*, s)\\ &= \sum_{\sigma\in\mathfrak{S}_n} \mathcal{F}(\varphi)(\sigma(\lambda^*), \sigma(s)) \cdot \Phi_{K,\sigma}(\lambda^*, s)\end{aligned}$$

Now, we compute $(\mathrm{Ch}_n(f) \circledS \mathrm{Ch}_{n''}(f''))(\lambda^*, s)$. If $f' = *_{1^{n'}}(\varphi')$ and $f'' = *_{1^{n''}}(\varphi'')$, by definition of shuffle algebra, Proposition 7.5, Proposition 6.16, Corollary 7.11, and Lemma 7.9,

$$
\begin{aligned}
(\mathrm{Ch}_{n'}(f') \circledS \mathrm{Ch}_{n''}(f''))(\lambda^*, s) &= \sum_{w \in \mathrm{Sh}(w,w')} (\mathrm{Ch}_{n'}(f') \otimes \mathrm{Ch}_{n''}(f''))(w^{-1}(\lambda^*), w^{-1}(s)) \cdot \Phi_{K,w^{-1}}(\lambda^*, s) \\
&= \sum_{w \in \mathrm{Sh}(w,w')} (\mathcal{F}(\widetilde{\mathrm{CT}}_{n'}(f')) \otimes \mathcal{F}(\widetilde{\mathrm{CT}}_{n''}(f'')))(w^{-1}(\lambda^*), w^{-1}(s)) \cdot \Phi_{K,w^{-1}}(\lambda^*, s) \\
&= \sum_{w \in \mathrm{Sh}(w,w')} \left(\mathcal{F}\left(\sum_{w' \in \mathfrak{S}_{n'}} M_{w'}(\varphi') \right) \mathcal{F}\left(\sum_{w'' \in \mathfrak{S}_{n''}} M_{w''}(\varphi'') \right) \right) (w^{-1}(\lambda^*), w^{-1}(s)) \cdot \Phi_{K,w^{-1}}(\lambda^*, s) \\
&= \sum_{w \in \mathrm{Sh}(w,w')} \Phi_{K,w^{-1}}(\lambda^*, s) \\
&\quad \cdot \sum_{\substack{w' \in \mathfrak{S}_{n'} \\ w'' \in \mathfrak{S}_{n''}}} \mathcal{F}(M_{w'}(\varphi'))(w^{-1}(\lambda^{*\prime}), w^{-1}(s'))\mathcal{F}(M_{w''}(\varphi''))(w_{-1}(\lambda^{*\prime\prime}), w^{-1}(s'')) \\
&= \sum_{w \in \mathrm{Sh}(w,w')} \Phi_{K,w^{-1}}(\lambda^*, s) \\
&\quad \cdot \sum_{\substack{w' \in \mathfrak{S}_{n'} \\ w'' \in \mathfrak{S}_{n''}}} \mathcal{F}(\varphi')(w'w^{-1}(\lambda^{*\prime}), w'w^{-1}(s'))\mathcal{F}(\varphi'')(w''w^{-1}(\lambda^{*\prime\prime}), w''w^{-1}(s'')) \\
&\quad \cdot \Phi_{K,w'}(w^{-1}(\lambda^{*\prime}), w^{-1}(s'))\Phi_{K,w''}(w^{-1}(\lambda^{*\prime\prime}), w^{-1}(s'')) \\
&= \sum_{w \in \mathrm{Sh}(w,w')} \Phi_{K,w^{-1}}(\lambda^*, s)\Phi_{K,w' \times w''}(w^{-1}(\lambda^*), w^{-1}(s)) \\
&\quad \cdot \sum_{\substack{w' \in \mathfrak{S}_{n'} \\ w'' \in \mathfrak{S}_{n''}}} \mathcal{F}(\varphi)((w' \times w'')w^{-1}(\lambda^*), (w' \times w'')w^{-1}(s)) \cdot \\
&= \sum_{w \in \mathrm{Sh}(w,w')} \sum_{\substack{w' \in \mathfrak{S}_{n'} \\ w'' \in \mathfrak{S}_{n''}}} \mathcal{F}(\varphi)((w' \times w'')w^{-1}(\lambda^*), (w' \times w'')w^{-1}(s)) \cdot \Phi_{K,(w' \times w'')w^{-1}}(\lambda^*, s) \\
&= \sum_{\sigma \in \mathfrak{S}_n} \mathcal{F}(\varphi)(\sigma(\lambda^*), \sigma(s)) \cdot \Phi_{K,\sigma}(\lambda^*, s)
\end{aligned}
$$

which agrees with our computation for $\mathrm{Ch}_n(f' * f'')(\lambda^*, s)$. Hence, $\mathrm{Ch} : SH \to \mathcal{SH}(\Phi_K)_{\mathcal{PW}}$ is an isomorphism, as desired.

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